Rules without Commitment: Reputation and Incentives*

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Abstract

This paper studies the optimal design of rules in a dynamic model when there is a time inconsistency problem and uncertainty about whether the policy maker can commit to follow the rule ex-post. The policy maker can either be a commitment type, which can always commit to enforce regulation, or an optimizing type which sequentially decides whether to enforce or not. This type is unobservable to private agents who learn about it through the actions of the policy maker. Higher beliefs that the policy maker is the commitment type (the policy maker's reputation) helps promote good behavior by private agents. We show that in a large class of economies late revelation of the policy maker's type is preferable from an ex-ante perspective. Therefore, learning the type of the policy maker can be costly relative to the case in which there is uncertainty about this type. If the initial reputation is not too high, the optimal rule is the strictest one that is incentive compatible for the optimizing type. We show that reputational considerations imply that the optimal rule is *more lenient* than the one that would arise in a static environment. Moreover, *opaque* rules are preferable to transparent ones.

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1 Introduction

Since Kydland and Prescott (1977), a large literature in macroeconomics has grappled with the problem of designing policy when there are time inconsistency problems. Rules are often proposed as a solution to the time inconsistency problem. The implicit assumption is that society can credibly impose rules on policy makers and that policy makers can commit to follow these rules. However, at the time when rules and regulations are formulated, there is often substantial *uncertainty* about whether policy makers can resist the temptation to deviate ex-post from the stated rules if it is optimal. This uncertainty is only resolved over time as the actions of policy makers are observed. The combination of uncertainty and learning generates reputational incentives for policy makers.

The key question motivating this paper is how should rules be designed taking into account both the uncertainty about the policy maker's ability to follow the rules ex-post and their reputational building incentives. To do this we study the optimal design of policy rules in a dynamic game between policy makers and private agents in which the commitment ability of the policy maker is private information. We define the public beliefs about the ability of the policy makers to commit as the policy maker's *reputation*. The main result of our paper is that if the initial reputation is low enough, optimal policy should be designed to preserve uncertainty in future periods. This is implemented by introducing leniency in policy. In contrast, if the reputation is high, optimal policy should promote learning about this type. We also show that designing opaque rules can be beneficial since they help preserve uncertainty especially when reputation is high.

The insights from our theory can be applied to many relevant policy design questions including: the design of central bank mandates, fiscal rules in a federal governments, and financial regulation. Consider, for instance, the optimal design of financial regulation. As is well understood, if regulators can commit, a no-bailout policy is a optimal in order to prevent excessive risk taking by financial institutions ex-ante. In particular, creditors should be forced to take losses in the event of default (bail-in). If the reputation of the regulators is not sufficiently high, our analysis suggests that allowing for partial bailouts in equilibrium is optimal. Contrary to conventional wisdom, we show that bailouts along the equilibrium path are necessary to discipline future risk-taking of financial firms as they preserve uncertainty about the type of the policy maker.

We consider a dynamic model with three types of agents: a rule designer, policy makers, and private agents. The rule designer chooses a rule, which consists of a policy recommendation to policy makers, in order to maximize expected social welfare. After the rule is chosen, the private agents take their actions and finally the policy maker chooses a policy. As in Barro (1986), the policy makers can be one of two types: a commitment type that always follows the recommendation or an optimizing type that follows the recommendation only if it is sequentially optimal. This type is unobservable to both the rule designer and private agents. We define the beliefs that the policy maker is the commitment type as its reputation.

We present two leading examples of our framework. The first is a model similar to Barro and Gordon (1983b) in which the rule designer must choose the optimal inflation target. The second is a banking model in the spirit of Kareken and Wallace (1978) where there is a tradeoff between providing incentives to bankers for taking appropriate levels of risk ex-ante and the bailing them out ex-post to avoid costly default. In this case the rule designer chooses an optimal bailout policy.

We first study a static problem. Since there is no way to incentivize the optimizing type to choose any policy other than the ex-post optimal one, the best the rule designer can do is to get the commitment type to follow the Ramsey policy. We show that under certain conditions, *uncertainty is beneficial* in that the expected social welfare is higher when the rule designer and private agents are uncertain about the type of the policy maker relative to the case in which types are revealed right before the rule designer chooses the rule. That is, the rule designer's static value is concave in the policy maker's reputation. The reason for this is that uncertainty about the policy maker's type plays a positive role in providing incentives to private agents.[1] In particular, under our conditions, private agents take actions closer to the Ramsey outcome when they are uncertain about the type of the policy maker they are facing.

We then consider a repeated version of this policy game. Unlike the static model, the optimizing type now cares about its reputation in the following period as it affects the actions of the private agents. Thus it can be incentivized to choose policies other than its static best response. We show that when reputation is low, the rule designer wants to preserve uncertainty about the type of the policy maker. The optimal rule is the most stringent policy that is incentive compatible for the optimizing type. This optimal recommended policy is more lenient than the statically optimal one. Leniency in the rule makes it easier for the optimizing type to follow the recommendation ex-post. This has dynamic benefits because it prevents private agents from learning the type of the policy maker and uncertainty is beneficial. When reputation is low, inducing the optimizing type to follow the rule also has static benefits. This is because it promotes better behavior by the private agents who anticipate that the optimizing type will follow the rule – albeit more lenient – instead of the statically optimal policy.

In contrast, if reputation is sufficiently high, the rule designer finds it optimal to design

[1] Nosal and Ordoñez (2016) also consider an environment in which uncertainty can mitigate the time inconsistency problem. The mechanism is very different: here there is uncertainty about the policy maker's type, while in their paper there is uncertainty about the state of the economy, which restrains the policy maker ex-post.

rules so that the type of the policy maker is revealed. This is because when reputation is sufficiently high there are static costs associated with choosing a lenient rule. In this case, private agents anticipate that the rule will be followed with sufficiently high probability and so by choosing the Ramsey policy, the rule designer can obtain a value close to the Ramsey outcome. There are however dynamic losses associated with choosing the Ramsey policy: if the rule is to follow the Ramsey policy, for a low enough discount factor, the optimizing type will not follow the rule and there will be revelation about the type of the policy maker in the first period. Because uncertainty is beneficial, the expected continuation value is lower than in the case in which the type of the policy maker is not revealed. When reputation is high enough, the static benefits of choosing a stringent rule outweigh the dynamic losses.

We then investigate other features of rule design that can help to maintain reputation without the static costs. In particular, we study the optimal degree of transparency of the rule. We say that a rule is transparent if deviations by the policy maker are easily detectable. In repeated policy games with no reputational considerations, perfect monitoring is always desirable. See Atkeson and Kehoe (2001), Atkeson et al. (2007), and Piguillem et al. (2013). In contrast, we show that with reputational considerations, transparent rules are desirable only for low levels of reputation while opaque rules are desirable for high levels of reputation.[2]

We consider two ways in which the rule designer can affect the transparency of the rules. First, we assume that future private agents and rule designers observe only a signal of the chosen policy and the rule designer chooses the precision of the signal. High precision (transparency) is beneficial because it incentivizes the optimizing type to follow the rule as a deviation results in the revelation of its type with large reputation losses. Low precision (opaqueness) is beneficial because it allows the rule designer to maintain uncertainty about the policy maker's type. For instance, if signals are imprecise, private agents attribute observed deviations from the stated policy to noise rather than the policy maker being the optimizing type that deviated. This is helpful for high level of reputation since the rule designer would like to choose the Ramsey policy from a static perspective. As discussed earlier, there is a trade-off between static value of having the commitment type follow a tough rule and the dynamic losses associated with learning the policy maker's type. Allowing for opaque rules helps break this trade-off: the rule designer can achieve both the high static pay-off of choosing a rule equal to the Ramsey policy without the costs associated with separation for sure because the policy observations are very noisy. We show it is optimal to choose a perfectly uninformative signal when the prior reputation is high enough. A similar argument implies that it is optimal to have short tenure for

[2] In the principal-agent literature there are examples of environments where imperfect monitoring is beneficial to provide incentives. See for instance Crémer (1995) and Prat (2005).

the policy maker when reputation is high.

An alternative way of introducing opacity in rules is to allow the rule designer to choose stochastic rules even though fundamentals are deterministic. When reputation is low, the optimal rule has no randomization in order to maximize the incentives of the optimizing type to follow more stringent policies. When reputation is high instead, it is optimal to have randomization in order to reduce the dispersion in the posteriors.

In our baseline setup, we model the commitment type as a policy maker that cannot deviate from the rules. One interpretation of this is that the commitment type suffers a cost from deviating from the stated rule over and above the reputational cost in the model. For example, a deviation may affect the commitment's type ability to be elected to higher offices while the optimizing type may not have such ambitions. Alternatively, one could assume that policy makers are identical but there is uncertainty about whether these policy deviations can be enacted, due to legislative holdups, for example. In particular, policy makers always have an incentive to choose policies which are sequentially rational, but might face roadblocks in implementation if the legislature is controlled by opponents who might block these policies for purely political purposes. As in Piguillem and Riboni (2018), the rule can be the default option in case of such disagreements. In this case, we can interpret the commitment type as a policy maker which faces such roadblocks and the optimizing type as one which does not. The latter might want to pretend as if its hands are tied (like the commitment type) for exactly the same reasons as in the baseline model.

An alternative approach is to assume that the two types of policy makers differ in their preferences (payoff types). For example, policy makers can differ in their temptation to deviate ex-post because certain policy makers can better resist the pressure from interest groups ex-post or simply they have difference preferences over outcomes than the social welfare function, as in the seminal Rogoff (1985) paper. The outcomes in this case differ from the ones in the baseline model: we show that with preference types and a reasonable belief refinement, the equilibrium with payoff types has separation for all levels of initial reputation.

Related literature This paper is related to the literature that studies the trade-off between rules and discretion. See for example Athey et al. (2005), Halac and Yared (2014), Halac and Yared (2017), and Azzimonti et al. (2016) among others. The focus of this literature is on how much flexibility to leave the policy maker when it is not possible to make the rule contingent on the state of the economy (say because it is private information to the policy maker). We abstract from this issue by considering a deterministic environment but we focus instead on the uncertainty about the ability of the policy maker to commit. Our paper is also related to the literature that studies optimal policies without commitment when it is known that the policy maker cannot commit. This is the approach followed by a large literature on time consistent policies, including, Barro and Gordon (1983b), Chari and Kehoe (1990), Phelan and Stacchetti (2001), andHalac and Yared (2018). Our paper nests simple versions of these two approaches as special cases when reputation is either one or zero.

This paper builds on the reputation literature that originates with Milgrom and Roberts (1982) and Kreps and Wilson (1982). See Barro (1986), Backus and Driffill (1985), and Phelan (2006), Amador and Phelan (2018), Dovis and Kirpalani (2018) for recent applications to policy games. Most of this literature takes as given the policy chosen by the commitment type and analyses the incentives of the optimizing type and the outcomes that can be achieved. The goal of this paper is to study the optimal policy that the commitment type should follow.

A key driver of our results is the idea that uncertainty about the policy maker type is beneficial. This feature is also present in Dovis and Kirpalani (2017). Our contribution is to show how this property affects the design of the optimal rule. Relatedly, Marinovic and Szydlowski (2019) consider an economy where uncertainty is beneficial in certain circumstances and study whether it is optimal to resolve uncertainty by credibly revealing the type. The counterpart of the rule designer in their model knows the type of the policy maker and they focus purely on information revelation. In contrast, the rule designer in our model does not know the policy maker's type and we focus on the design of policies that can induce – or not – revelation.

Our paper is also related to a literature that studies signaling games when policy makers have different types. See for instance Vickers (1986), Cole et al. (1995), and Angeletos et al. (2006) with payoff types or Dovis and Kirpalani (2017) where one type has the ability to commit to the announced policy. See also Sanktjohanser (2018) for a similar analysis in the context of a bargaining game. Our approach differs from these papers since we study the best policy chosen by the rule designer when there is uncertainty about the type of the policy maker while these papers study what is the optimal policy that the commitment type would choose knowing its type. We show that if the rules are chosen by the policy maker, the commitment type (if sufficiently patient) chooses a stringent rule to separate from the optimizing type for all levels of reputations while the rule designer under the veil of uncertainty chooses to avoid separation when the reputation is sufficiently low.

Debortoli and Nunes (2010) consider a policy game in which the policy maker has the ability to change its polices infrequently and randomly. They abstract from reputation building incentives.

2 Policy Game

We consider a policy game that captures a variety of relevant economic environments as special cases. We present two leading examples of our framework: a version of the Barro and Gordon (1983a) model of monetary policy and a banking model in the spirit of Kareken and Wallace (1978). Our framework also nests other models, including the Fisher model of capital income taxation considered in Chari and Kehoe (1990) and the model of bailouts in a federal government in Dovis and Kirpalani (2017).

There are three types of agents: the *rule designer*, *policy makers* (or bureaucrats), and a continuum of private agents. We consider a repeated environment where there are no endogenous state variables across periods. At the beginning of each period, the rule designer recommends a policy π_r from a set $[\underline{\pi}, \overline{\pi}]$. We refer to this recommendation as a *rule*. Private agents then choose an individual action. After observing the private action, the policy maker chooses a policy π. The policy maker can be one of two types: a *commitment* type that always follows the recommendation made by the rule designer, or an *optimizing* type which can choose any policy π in the set $[\underline{\pi}, \overline{\pi}]$. The policy maker's type is unobservable to the private agents and the rule designer, who learn about it through the observed policies. We assume that the private agents and the rule designer share the same prior ρ that they are facing the commitment type.

We let x denote the representative (average) action taken by private agents. We assume that the private action is a function ϕ of the expected policy, $\mathbb{E}\pi = \rho\pi_c + (1-\rho)\pi_o$ where $\pi_c = \pi_r$ is the policy chosen by the commitment type and π_o is the policy implemented by the optimizing type,

$$x = \phi\left(\mathbb{E}\pi\right). \tag{1}$$

We will refer to (1) as the *implementability constraint*. We think of the function ϕ as summarizing the set of implementability conditions describing the set of outcomes that can be implemented given a set of policies or an incentive compatibility constraint.

The rule designer and the policy makers maximize a social welfare function $w(x, \pi)$. We assume that the problem is time inconsistent. Specifically, we define the Ramsey outcome as

$$(x_{ramsey}, \pi_{ramsey}) = \arg\max_{x,\pi} w(x,\pi) \quad \text{subject to} \quad x = \phi(1,\pi)$$

We assume that the Ramsey policy is not optimal ex-post in that $\pi_{ramsey} \neq \pi^*(x_{ramsey})$ where $\pi^*(x)$ denotes the best response of the government to x, $\pi^*(x) = \arg\max_\pi w(x,\pi)$. We assume without loss of generality that $\pi^*(x_{ramsey}) > \pi_{ramsey}$.

We also make the following assumptions about w and ϕ:

Assumption 1. *Assume that*

1. *If* $w_x > 0$ *then* $\phi' \leqslant 0$, $\phi'' \leqslant 0$, *and* $w_{x\pi} < 0$

2. *If* $w_x < 0$ *then* $\phi' \geqslant 0$, $\phi'' \geqslant 0$, *and* $w_{x\pi} > 0$.

As is standard in the time-inconsistency literature, we consider environments in which the the inability of the policy maker to commit ex-post incentivizes private agents to take worse actions ex-ante. Thus, if social welfare is increasing in the private action x, we assume that if agents expect higher π, they choose lower values of x ($\phi' \leqslant 0$). We also assume that the private action x is concave in expected policy. Finally, we assume a form of supermodularity in (x, π) which implies that the government's incentive to deviate from its ex-ante promises is higher, the worse the private action (low x) is.

We next present two economies and show how they map into our general framework.

Example 1: Barro-Gordon One special case of the general environment is the classic Barro and Gordon (1983a) model used to analyze the time inconsistency problem in monetary policy. In this context, we interpret x as the average wage inflation and π is the money growth rate (or price inflation).

We assume that private agents set wage inflation according to

$$x = \phi(\mathbb{E}\pi) = [\rho\pi_c + (1-\rho)\pi_o].$$

The social welfare function takes the quadratic form

$$w(x, \pi) = -\frac{1}{2}\left[(\psi + x - \pi)^2 + \pi^2\right]$$

with $\psi > 0$. The first term in this functional form represents the welfare losses associated with low employment due, for example, to monopolistic competition in labor markets. The parameter ψ measures the extent of this distortion and it can be mapped into the wage markup set by unions. The second term captures the costs of ex-post inflation (due, for example, to the transactional value of real money balances).

Example 2: Bailout and effort We now consider another economy inspired by the classic analysis in Kareken and Wallace (1978) which studies the trade-off between the ex-post benefits and the ex-ante costs of bailouts.

There are two types of private agents: depositors and bankers. At the beginning of each period, the banker must borrow $k = 1$ from the depositors to finance an investment opportunity that pays off at the end of the period. The return of the investment opportunity is R_H with probability $p(e)$ where e is the effort exerted by the banker and 0 with probability $1 - p(e)$. The function $p(\cdot)$ is increasing and concave and it satisfies Inada conditions. Exerting effort e results in a utility cost $\nu(e)$ where ν is increasing and convex. We interpret the effort as the costs associated with monitoring the investment project. Bankers and depositors are risk-neutral and do not discount consumption between the beginning and the end of the period.

The banker offers a contract to depositors that promises to repay R units of the consumption good in the second sub-period subject to limited liability. We assume that society faces bankruptcy costs ψ whenever the lenders recover less than their initial investment.[3] The policy maker can avoid these bankruptcy costs by making a transfer to the banker to enable him to repay the depositors. In particular, the government can choose the recovery π in case the banker is unable to repay. There is a taxation cost associated with these transfers, denoted by $c(\pi)$ where c is increasing and convex. To simplify calculations we assume that $p(e) = e^{\alpha}$, $\nu(e) = e^2/2$, and $c(\pi) = \lambda\pi^2/2$. We assume that if the recovery is π, the bankruptcy costs are $\psi(1-\pi)$.

We assume that depositors can observe the effort e. Depositors are then willing to lend to the banker if the interest rate is at least

$$R(e) = \frac{1-(1-p(e))\left[\rho\pi_c + (1-\rho)\pi_o\right]}{p(e)} \tag{2}$$

The banker chooses effort to maximize $-\nu(e) + p(e)\left[R_H - R(e)\right]$ subject to (2). Using (2) to substitute for $R(e)$, we can rewrite the banker's problem as

$$\max_e -\nu(e) + p(e) R_H + (1-p(e))\,\mathbb{E}\pi$$

where the term $(1-p(e))\,\mathbb{E}\pi$ represents the distortion induced by the expected bailout. Thus the optimal effort e is a function $\phi(\mathbb{E}\pi)$ that is implicitly defined by the first order condition

$$\nu'(e) = p'(e)\left[R_H - \mathbb{E}\pi\right]$$

The social welfare function is the equally weighted sum of the utility of the bankers and depositors net of taxation and bankruptcy costs

$$w(e,\pi) = -\nu(e) + p(e) R_H - 1 - (1-p(e))(1-\pi)\psi - c(\pi).$$

3 Optimal Rules

In this section, we consider the problem of how to design the optimal rule. We begin by characterizing the rule designer's problem in a static setting and next study how the op-

[3] Alternatively, we could have assumed that these costs are suffered whenever lenders recover less than the promised return.

timal rule changes once we introduce dynamics. We also study the case in which the rule designer is unable to choose the rule each period. Finally, we study how the conclusions change if we allow the rule to be chosen by policy makers instead.

3.1 Statically Optimal Rules

We begin by studying the optimal rule in a static setting. The rule designer anticipates that if the policy maker is the commitment type, it will follows the rule π_r. Instead, if the policy maker is the optimizing type, it will always choose the static best response to the private action x. This is because, in a static model, the rule designer has no tools to incentivize the optimizing type to take any other action. Of course, this will change in the dynamic setting.

We can then write the problem for the rule designer as choosing the recommendation for the commitment type, π_c, to solve

$$W_0(\rho) = \max_{\pi_c} \rho w(x, \pi_c) + (1-\rho) w(x, \pi_o) \tag{3}$$

where given π_c and the prior ρ, x and π_o are given by

$$\begin{aligned} x &= \phi(\rho\pi_c + (1-\rho)\pi_o) \\ \pi_o &= \pi^*(x) \end{aligned} \tag{4}$$

For later reference, we denote the solution to this problem as $\pi_{c0}(\rho)$, $\pi_{o0}(\rho)$, and $x_0(\rho)$. We can also define the value for the optimizing type:

$$V_0(\rho) = w(x_0(\rho), \pi^*(x_0(\rho))).$$

We next discuss conditions under which*uncertainty is beneficial* in that

$$W_0(\rho) \geqslant \rho W_0(1) + (1-\rho) W_0(0). \tag{5}$$

When uncertainty is beneficial, the expected social welfare is higher when the policy maker type is uncertain relative to the case in which types are revealed right before the rule designer chooses the rule. This property of the static problem turns out to be critical for the form of the optimal rule in a dynamic model.[4]

We nowe provide a set of sufficient conditions on primitives that ensure that uncertainty is beneficial. In the Appendix we show that the Barro-Gordon and bailout example satisfy these assumptions.

[4]In Appendix A.10 we provide an example of an environment that does not satisfy this property.

Assumption 2. *Assume that*

1. $w(x,\pi)$ *is concave in* (x,π)

2. $w_\pi(x,\pi)$ *is convex in* (x,π)

3. $1 > \pi^*{}_x(x)\,\phi'(\pi) \geqslant 1 - \frac{w_x(\underline{x},\pi^*(\underline{x}))}{w_x(\underline{x},\underline{\pi})}$ *where* $x = \phi(\pi)$, $\underline{x} = \phi(\underline{\pi})$, *and* $\pi^*{}_x(x) = -\frac{w_{x\pi}(x,\pi^*(x))}{w_{\pi\pi}(x,\pi^*(x))}$.

4. $w_\pi(x,\pi) + [\rho w_x(x,\pi) + (1-\rho)\, w_x(x,\pi^*(x))]\, \frac{\phi'(\cdot)}{[1-\phi'(\cdot)(1-\rho)\pi^*_x(x)]} \leqslant 0$ *for all* π.

We have the following lemma:

Lemma 1. *Under Assumptions 1 and 2*, $\pi_{c0}(\rho) = \underline{\pi}$ and uncertainty is beneficial in that (5) holds..

The optimal static rule takes a simple form: for all ρ, the rule is set to $\pi_c(\rho) = \underline{\pi}$ which is also the Ramsey policy.[5] This result follows from Condition 4 in Assumption 2. The expression in Condition 4 is the first order condition for problem in (3). The condition implies that reducing π has a positive marginal effect and thus it is optimal to be at the corner $\underline{\pi}$. Therefore, it is optimal for the rule designer to recommend the toughest/strictest possible rule. In the context of the Barro-Gordon example, this says that the optimal inflation target is zero while in the bailout example, a strict no-bailout policy is optimal.

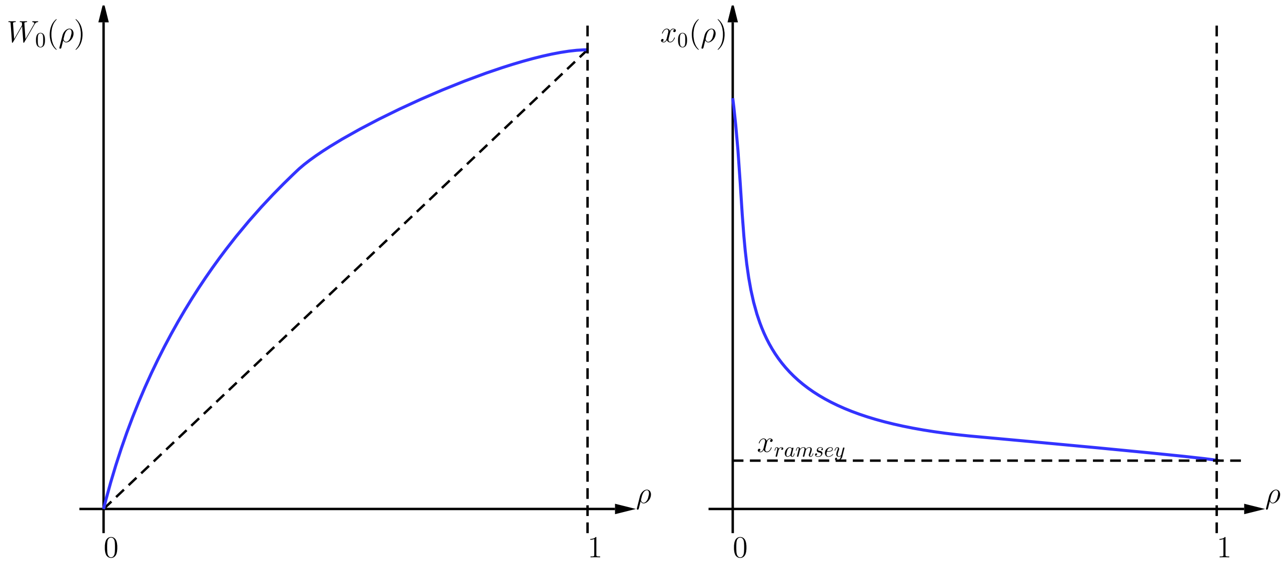
We now show that under our assumption, condition (5) holds. Intuitively, uncertainty is beneficial if for interior levels of reputation, the losses associated with moving from the average private action $x_0(\rho)$ to the worst outcome $x_0(0)$ is larger than the gains associated with moving to the Ramsey outcome $x_0(1)$. A critical step to show that this is to establish that $x_0(\rho)$ is convex (concave) when $w_x < 0$ ($w_x > 0$) as shown Figure 1.

To establish that uncertainty is beneficial it is sufficient to show that $W_0(\rho)$ is concave. We illustrate the logic of the proof for the case in which $w_x < 0$ – like in the Barro-Gordon model. A specular logic holds for the case in which $w_x > 0$. Concavity of w – Condition 1 – implies that $w(x_0(\rho),\underline{\pi})$ and $w(x_0(\rho),\pi_o(\rho))$ are concave in ρ if x_0 is convex. The convexity of x_0 is guaranteed by Conditions 1 and 2. To see this, note that $x_0(\rho) = \phi(\rho\underline{\pi} + (1-\rho)\,\pi_{o0}(\rho))$. Intuitively, since ϕ is convex from Assumption 1, x_0 is convex if π_{o0} is decreasing and convex in ρ. Condition 2 guarantees that π_{o0} is convex.

Having established the concavity of $w(x_0(\rho),\underline{\pi})$ and $w(x_0(\rho),\pi_o(\rho))$ is not enough to show that $W_0(\rho) = \rho w(x_0(\rho),\underline{\pi}) + (1-\rho)\, w(x_0(\rho),\pi_o(\rho))$ is concave since the product of two concave functions is not necessarily concave. However the technical assumption in Condition 3 guarantees that $W_0(\rho)$ is concave.

[5] This is true even though the private action is not at the Ramsey level since private agents anticipate that

Figure 1: Static value and private action



3.2 Dynamic Problem

We now study the optimal rule design problem in a dynamic setting. We start by repeating the stage game, studied in the previous section, twice and then analyze what happens as the number of periods goes to infinity.

When there is more than one period, the optimizing type can be incentivized to take a different action from its static best response. We can set up the rule designer's problem as choosing the rule that will be followed by the commitment type, $\pi_r = \pi_c$, and a recommendation to the optimizing type, π_o. This recommendation must be incentive compatible in that the optimizing type must prefer to follow the recommendation than to choose its best possible deviation (playing the static best response $\pi^* (x)$) and attaining a continuation value $V_0(0)$ as the prior jumps to zero:

$$w(x, \pi_o) + \beta_o V_0\left(\rho'(\pi_o)\right) \geqslant w(x, \pi^*(x)) + \beta_o V_0(0) \tag{6}$$

where β_o is the discount factor for the optimizing type and $\rho'(\pi_o)$ is the private belief about the policy maker's type after observing π_o (given recommendation π_c). The law of motion for beliefs follows Bayes' rule

$$\rho'(\pi) = \begin{cases} \frac{\rho}{\rho+(1-\rho)\sigma} & \text{if } \pi = \pi_c \\ 0 & \text{o/w} \end{cases} \tag{7}$$

with probability $1-\rho$ the policy maker is the optimizing type who will deviate from the recommendation and choose the static best response.

where σ is an indicator variable that takes value 1 if private agents expect the optimizing type to choose the same policy as the commitment type, $\pi_o = \pi_c$, and $\sigma = 0$ otherwise.

For all subsequent analyses we assume that the policy makers are sufficiently impatient so that the Ramsey outcome is not incentive compatible:

Assumption 3. *The discount factor* β_o *is small enough so that*

$$w\left(x_{ramsey}, \pi^*\left(x_{ramsey}\right)\right) - w\left(x_{ramsey}, \pi_{ramsey}\right) > \frac{\beta_o}{1-\beta_o}\left[V_0\left(1\right) - V_0\left(0\right)\right]$$

We can then write the rule designer problem as

$$W\left(\rho\right) = \max_{x,\pi_c,\sigma} \rho\left[w\left(x,\pi_c\right) + \beta W_0\left(\rho'\left(\pi_c\right)\right)\right] + \left(1-\rho\right)\left[w\left(x,\pi_o\right) + \beta W_0\left(\rho'\left(\pi_o\right)\right)\right] \tag{8}$$

subject to the implementability condition,

$$x = \phi\left(\rho\pi_c + \left(1-\rho\right)\left[\sigma\pi_c + \left(1-\sigma\right)\pi^*\right]\right),$$

the incentive compatibility constraint for the optimizing type (6), and the law of motion for beliefs (7). Note that we allow in principle for the rule designer's discount factor β to differ from β_o, although this is not critical.

For simplicity we abstract from mixed strategies for the optimizing type. In Appendix A.2, we show that this is without loss of generality. Under our assumptions, the outcome in which the optimizing type follows the rule with probability $\sigma \in (0,1)$ and the ex-post optimal policy with probability $1-\sigma$ is dominated in terms of welfare by the best equilibrium in which the optimizing type follows the rule with probability one.

We can then reduce the problem above to a discrete choice between two options: separating or pooling. If the rule designer chooses to separate, it chooses the best static rule. Because of Assumption 3, the Ramsey outcome is not incentive compatible and the optimizing type will choose the static best response and not follow the rule so the type of the policy maker is revealed at the end of the period. Thus the continuation value is either $W_0\left(1\right)$ with probability ρ or $W_0\left(0\right)$ with probability $1-\rho$. The value of separating is then

$$W_{sep}\left(\rho\right) = W_0\left(\rho\right) + \beta\left[\rho W_0\left(1\right) + \left(1-\rho\right)W_0\left(0\right)\right]$$

If the rule designer chooses to pool, it sets the rule to $\pi_{1,ico}\left(\rho\right)$ which is the lowest policy π consistent with the incentive compatibility constraint for the optimizing type:

$$w\left(\phi\left(\pi_{1,ico}\left(\rho\right)\right), \pi_{1,ico}\left(\rho\right)\right) + \beta V_0\left(\rho\right) = w\left(\phi\left(\pi_{1,ico}\left(\rho\right)\right), \pi^*\left(\phi\left(\pi_{1,ico}\left(\rho\right)\right)\right)\right) + \beta V_0\left(0\right).$$

Both types of policy makers follow the rule in equilibrium and thus uncertainty about the type is preserved and the continuation value is $W_0(\rho)$. The value of pooling is then

$$W_{\text{pool}}(\rho) = w(\phi(\pi_{1,\text{ico}}(\rho)), \pi_{1,\text{ico}}(\rho)) + \beta W_0(\rho).$$

The next proposition shows that designing a rule that preserves uncertainty about the policy maker's type is valuable when reputation is low:

Proposition 1. *Under Assumptions 1–3,*

1. *For ρ close to one there is separation with probability 1 and $\pi = \pi_{0c}(\rho) = \underline{\pi}$;*

2. *For ρ close to zero there is pooling ($\sigma = 1$) and $\pi_c(\rho) > \pi_{0c}(\rho) = \underline{\pi}$.*

In particular, for the Barro-Gordon economy, the optimal regulation has a cutoff property in that there exists a $\rho_1^* \in (0,1)$ such that:

1. *For $\rho > \rho_1^*$ it is optimal to separate and $\pi = \pi_{0c}(\rho) = \underline{\pi}$;*

2. *For $\rho \leqslant \rho_1^*$ it is optimal to pool and $\pi_c(\rho) = \pi_{1,\text{ico}}(\rho) > \pi_{0c}(\rho) = \underline{\pi}$.*

The key implication of this Proposition is that in contrast to the static case, when reputation is low, the rule designer recommends more lenient rules in order to preserve uncertainty about the policy market's type in the future.

To see why this is indeed the case, consider

$$W_{\text{pool}}(\rho) - W_{\text{sep}}(\rho) = \Delta\omega(\rho) + \beta\Delta\Omega(\rho)$$

where $\Delta\Omega(\rho) \equiv W_0(\rho) - [\rho W_0(1) + (1-\rho) W_0(0)]$ are the dynamic benefits of pooling and $\Delta\omega(\rho)$ are the static benefits of pooling given by

$$\Delta\omega(\rho) \equiv w(\phi(\pi_{1,\text{ico}}(\rho)), \pi_{1,\text{ico}}(\rho)) - W_0(\rho).$$

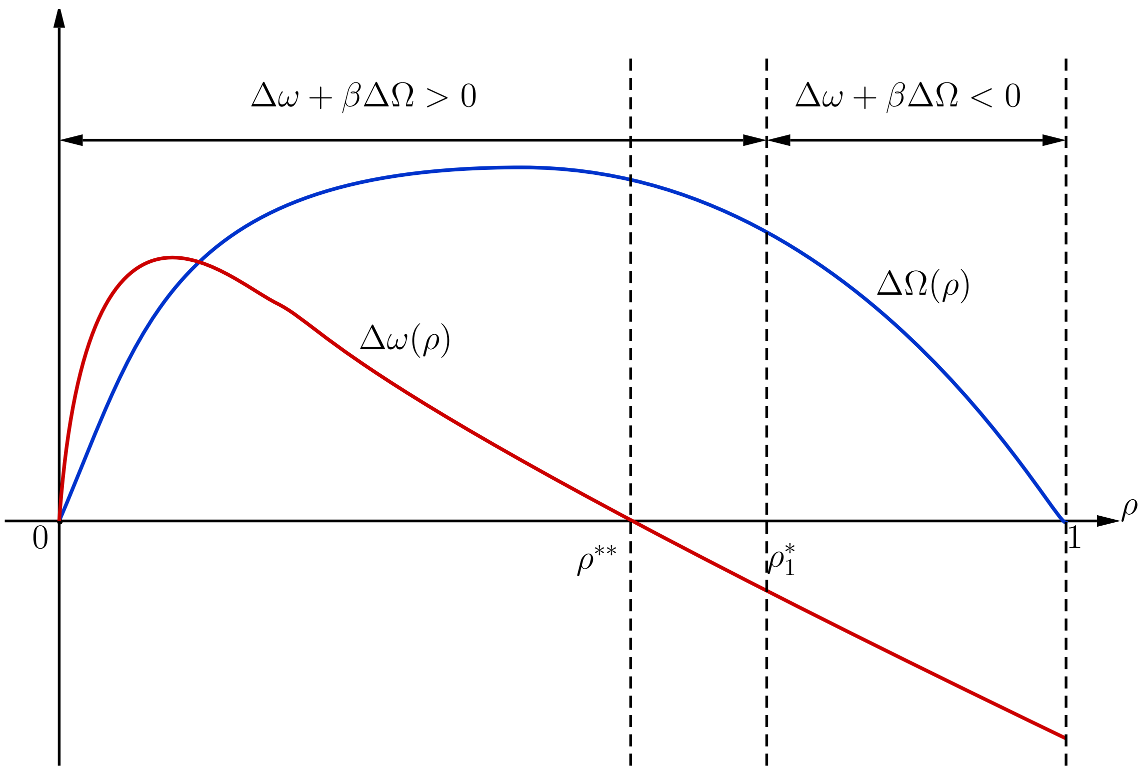
Since uncertainty is beneficial, we know that $\Delta\Omega(\rho) > 0$ for all $\rho \in (0,1)$ and equal to zero when there is no uncertainty and $\rho \in \{0,1\}$. Also by construction, the static benefits of pooling are zero for $\rho = 0$ since $\pi_{1,\text{ico}}(0) = \pi_{o0}(0) = \pi^*(x_0(0))$ and negative for $\rho = 1$ since $W_0(1)$ attains the Ramsey value and $w(\phi(\pi_{1,\text{ico}}(\rho)), \pi_{1,\text{ico}}(\rho)) < W_{\text{ramsey}}$ because the incentive constraint is assumed to be binding for all ρ (Assumption 3).

Combining these observations, it is immediate that for ρ close to one $W_{\text{sep}}(\rho) > W_{\text{pool}}(\rho)$ since the dynamic benefits are approximately zero and $\Delta\omega(\rho) < 0$. In the proof, we show that the static benefits of pooling $\Delta\omega(\rho)$ are increasing in ρ for low level of reputation. Intuitively, in the pooling regime the rule designer is inducing the optimizing type to follow a tougher policy than the static best response, $\pi_{1,\text{ico}}(\rho) < \pi^*(x_0(\rho))$

at the cost of forcing the commitment type to follow a laxer policy, $\pi_{1,ico}(\rho) > \underline{\pi}$. When reputation is low enough, this makes the expected policy tougher in the pooling regime as compared with the separating regime because in the latter, private agents expect the recommended policy to be followed with a low probability. Thus, pooling has both static and dynamic benefits and is therefore preferable.

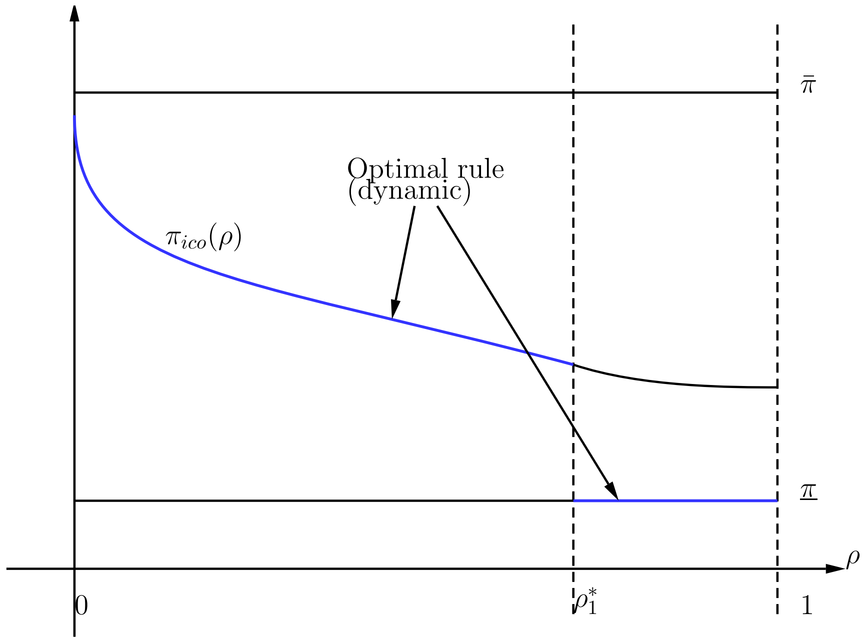
For our Barro-Gordon examples we can provide a tighter characterization of the optimal policy and show that it has a cutoff property. The proof for this is in the Appendix. (For the bailout example we verify that this is the case numerically.)

Figure 2: **Dynamic and static benefits of pooling**



Let's now consider what Proposition 1 implies for our two examples. In the bailout example, the optimal static rule is a strict no-bailout policy. However, in the dynamic model, on-path bailouts are necessary to achieve good outcomes when reputation is low. This is precisely because allowing for bailouts makes it easier for the optimizing type to follow the designer's recommendation and thus helps to preserve uncertainty going forward. This is beneficial because uncertainty about the policy maker's type prevents bankers from taking on excessive risk by exerting little effort. Similarly in the Barro-Gordon model, having looser inflation targets is beneficial when reputation is low.

Figure 3: **Optimal dynamic rule**



3.3 Limit of Finite Horizon

We now show that the insights from the two period model extend to any horizon. In particular, we analyze the limit of the finite horizon economy and show that an analog of Proposition 1 holds.

Let T be the horizon of the economy. For a fixed T, let $\{\pi_t^T(\rho)\}_{t=0}^T$ be the optimal rules set by the rules designer at each date. In the previous section we characterized the case for $T = 1$. We will use the following property:

Assumption 4. *The gains from going to best response are decreasing in* π*, in that*

$$G(\pi) \equiv w(\phi(\pi), \pi^*(\phi(\pi))) - w(\phi(\pi), \pi)$$

is monotone decreasing in π *(or* x*).*

This property is met in our two examples and it is satisfied in a general environment if an additional sufficient condition is satisfied as shown in the Appendix.

The next proposition shows that the limit of the finite horizon economy has a cut-off property: it is optimal to pool for priors below the cutoff while it is optimal to separate for priors above the cutoff:

Proposition 2. *Under Assumptions 1–4, as* $T \to \infty$*, there exists* ρ^* *such that*

1. *At* $\rho = 0$*, the outcome is the repetition of the static Markov outcome and* $\pi_c = \pi_0(0)$

2. *For* $\rho \in (0, \rho^*]$, *there is pooling in all periods and* $\pi_c(\rho) = \pi_{ico}$ *defined as*

$$w(\phi(\pi_{ico}), \pi_{ico}) = (1-\beta) w(\phi(\pi_{ico}), \pi^*(\phi(\pi_{ico}))) + \beta V_0(0) \tag{9}$$

3. *For* $\rho \in (\rho^*, 1]$, *there is separation in the first period and so* $\pi_c = \pi_0(\rho)$.

Qualitatively, the optimal rule is the same as in the two-period model. For low levels of reputation it is optimal to choose rules that do not reveal the type of the policy maker. Note that the optimal policy in the pooling regime does not depend on the prior ρ in the limit. This is because if it is optimal to pool today it is also optimal to pool in all subsequent periods. In this case, the type of the policy maker will never be revealed and so ρ does not affect the value on the equilibrium path. The initial prior also does not affect the value of the deviation on the right side of (9). This is because upon deviation the prior jumps to zero independently of the initial value. Thus the value of pooling is independent of ρ as shown in Figure 4. Since the value of the separating regime is increasing in ρ then the optimal rule has a cut-off property: pool if and only if ρ is below a cut-off ρ^*.

Further note that there is a discontinuity at $\rho = 0$. This is because when $\rho = 0$ it is not possible to incentivize the optimizing type to choose any policy other than its static best response.

Comparison with best PBE We now compare the the limit of the finite horizon to the best PBE (in the infinite horizon economy).

Proposition 3. *Under Assumption 3, the best PBE from the planner's perspective is such that it is always optimal to separate for all* $\rho > 0$. *The value is higher than the limit of the finite horizon.*

The value of the best equilibrium is plotted in Figure 4 and denoted by $\bar{W}(\rho)$. In the best PBE, it is always optimal to separate even when uncertainty is beneficial in the finite horizon economy. This is because trigger strategies can substitute for reputation and it is beneficial to use the commitment power of the commitment type. In fact, the value of the pooling regime equals the value of the best equilibrium when the rule designer knows that it is facing the optimizing type for sure, $W_{pool} = \bar{W}(0)$. This is because when $\rho = 0$, it is possible to support π_{ico} with trigger strategies to the worst equilibrium, $\underline{W}(0)$ which equals $W_0(0)/(1-\beta)$. Instead, in the limit of the finite horizon economy,once private agents learn that the policy maker is the commitment type, the only outcome that can be supported in the repetition of the static economy with $\rho = 0$ with value $W_0/(1-\beta) = \underline{W}(0)$.

Thus the value of separating in the best PBE is

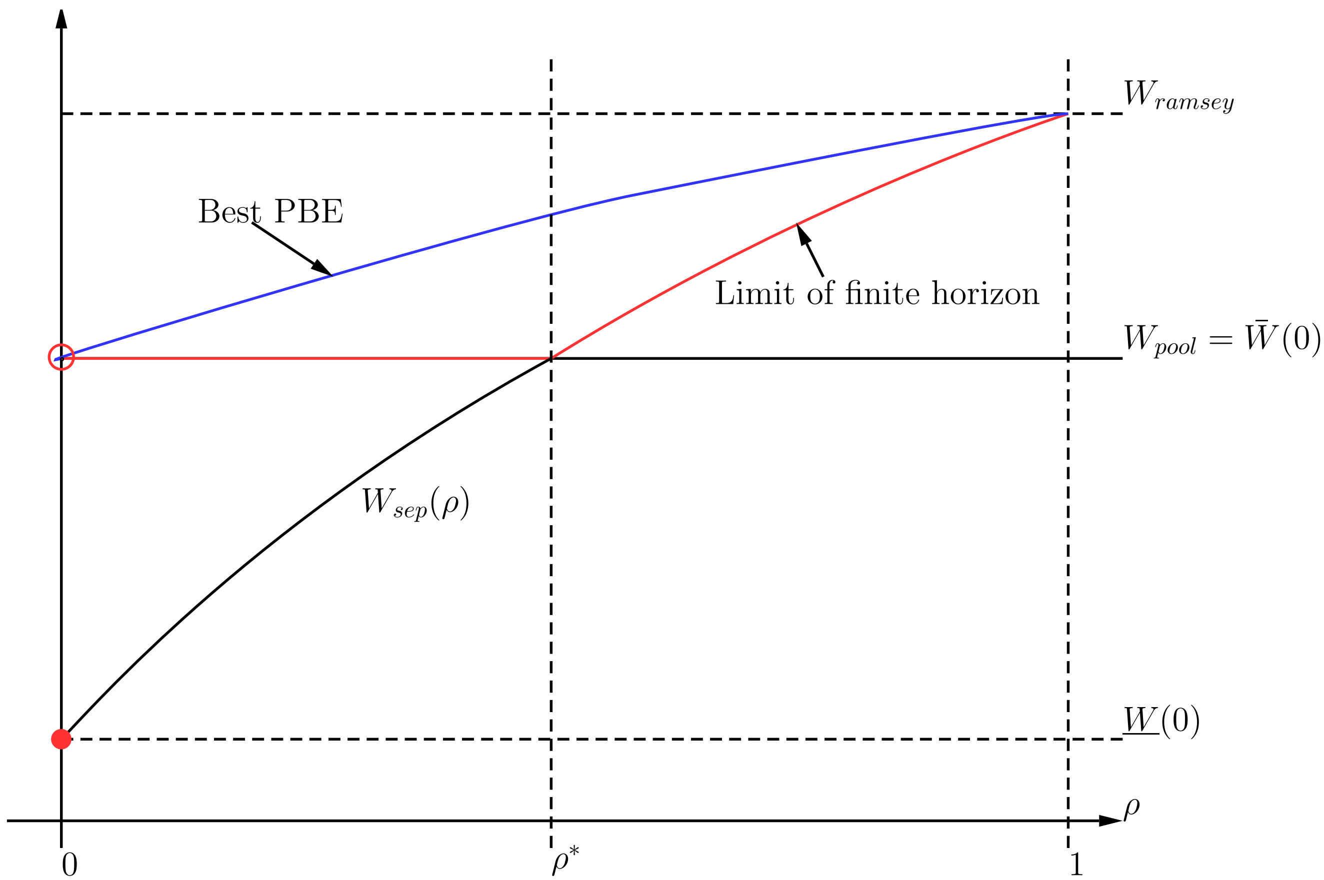
$$W_0(\rho) + \beta\left[\rho W_{ramsey} + (1-\rho)\bar{W}(0)\right] > W_{pool}$$

since $W_0(\rho)$ is greater than the static value of pooling, $\bar{W}(0) = W_{pool}$, and $W_{ramsey} > W_{pool}$. Thus it is optimal to separate in the best equilibrium. For comparison, the value of separating in the limit of the finite horizon is

$$W_0(\rho) + \beta\left[\rho W_{ramsey} + (1-\rho)\,\underline{W}(0)\right]$$

which is greater than W_{pool} only if ρ is sufficiently high since $\underline{W}(0) < W_{pool}$.

Figure 4: **Equilibrium Values**



3.4 Sticky Rules

So far we have allowed the rule designer to choose a new rule in each period as a function of the current reputation of the policy maker. In practice, opportunities for revising and introducing new rules arise infrequently. We now modify our framework to allow for this feature and show that our main conclusions are unchanged. In particular, the characterization in Proposition 2 continues to hold.

We consider the case in which rules are "sticky" in that they can only be changed in a given period with probability α. The analyses in the previous section considered the case in which $\alpha = 1$. We now assume that $\alpha < 1$. Let $\tilde{W}_{t+1}(\rho', \pi_c)$ be the rule designer's value next period if it cannot set a new rules and must use π_c and $\tilde{V}_{t+1}(\rho'_o, \pi_c)$ be the analogous value for the optimizing type. Fixing the horizon T, the problem for a rule designer that has the opportunity to set new rules in period $t < T$ can be written as

$$W_t(\rho) = \max_{x, \pi_c, \pi_o} \rho \left[w(x, \pi_c) + \beta\alpha W_{t+1}(\rho'_c) + \beta(1-\alpha)\tilde{W}_{t+1}(\rho'_c, \pi_c) \right] + (1-\rho)\left[w(x, \pi_o) + \beta\alpha W_{t+1}(\rho'_o) + \beta(1-\alpha)\tilde{W}_{t+1}(\rho'_o, \pi_c) \right]$$

subject to the implementability condition (4), the evolution of the prior (7), and the incentive compatibility constraint

$$w(x, \pi_o) + \beta\alpha V_{t+1}(\rho'_o) + \beta(1-\alpha)\tilde{V}_{t+1}(\rho'_o, \pi_c) \geqslant w(x, \pi_o) + \beta V_{t+1}(0). \tag{10}$$

Under our assumptions, if the rule designer wants to separate, its value is the same as the one in the previous section since the optimal rule under separation is $\underline{\pi}$ for all ρ and t. However, the introduction of sticky policies affects the rule designer's value when it chooses to pool. In particular, choosing π_c equal to the lowest value consistent with the optimizing type's incentive compatibility constraint induces the optimizing type to pool in the current period but it may induce separation in future periods if the rules cannot be adjusted. Thus the rule designer may want to choose an even laxer policy, which implies that (10) is slack, to ensure pooling next period in the event that the rule cannot be adjusted. However, this trade-off vanishes in the limit as the time goes to infinity since

$$\lim_{T\to\infty} \pi^T_{ico,t}(\rho) = \lim_{T\to\infty} \pi^T_{ico,t+1}(\rho) = \pi_{ico}$$

where π_{ico} is defined in (9). Thus the minimal rule that ensures pooling is constant over time. This observation implies that Proposition 2 holds with sticky policies ($\alpha < 1$).

3.5 Comparison to a Signaling Game

We next argue that the optimal rule characterized in the previous sections differs from the outcome of a signaling game in which the rule is chosen by the policy maker (that knows its type) instead of the rule designer who is uncertain about the type of the policy maker. If the rules are chosen by policy makers, the commitment type (if sufficiently patient) will choose a rule that induces separation for *all* levels of reputation. In particular, if will prefer to separate for low levels of reputation even though the rule designer strictly prefers to pool. This result mirrors the one in Dovis and Kirpalani (2017).

Proposition 4. *Under Assumptions 1 and 3, the outcome of the signaling game is such that the* *commitment and optimizing types announce different rules if either ρ is sufficiently high or ρ is small and $\beta_o = \beta_c \in (\underline{\beta}, \bar{\beta})$. Thus, in both cases there is separation after one period.*

The main idea here is that there are no dynamic gains for the commitment type of preserving uncertainty. In a separating equilibrium, the continuation value is always higher from the commitment type's perspective because since private agents know that they are facing the commitment type, it can implement the Ramsey outcome. However there may still be static benefits of pooling when reputation is sufficiently low as we saw in Section 2.2. However, if the discount factor is sufficiently high ($\beta > \underline{\beta}$), the dynamic benefits outweigh the static losses. Note that for this to be an equilibrium we also need the optimizing type to strictly prefer to separate which requires the discount factor to be low enough ($\beta < \overline{\beta}$). We show that $\underline{\beta} < \overline{\beta}$ since the optimizing type has additional static benefits of separating owing to the fact that it can choose its policy after private agents have chosen their action.

3.6 Payoff Types

So far, we have modeled the commitment type as a policy maker that cannot deviate from the rule. An alternative to modeling the uncertainty about the policy maker's ability to follow the rule is to assume that the two types of policy makers differ in their preferences. In particular, policy makers can differ in their temptation to deviate ex-post because certain policy makers can better resist to pressure from interest groups ex-post or if they have different preferences over outcomes than the social welfare function, as in the seminal Rogoff (1985) paper.

We next show that with preference types and a reasonable refinement requirement we have different outcomes than in our benchmark case. In particular, the equilibrium coincides with the outcome of the signaling game and there is separation for all levels of initial reputation.

We make our point in the context of the bailout example. Recall that the social welfare function is

$$w(x, \pi; \psi) = -\nu(x) + p(x) R_H - \psi(1 - p(x))(1 - \pi) - c(\pi)$$

where x is the banker's effort given by $\phi(\mathbb{E}\pi)$ for some ϕ with $\phi' < 0$, $\phi'' > 0$, $p(x)$ is the probability that the investment succeed, and $\psi(1 - p(x))(1 - \pi)$ is the default cost that can be mitigated by transfers π. The parameter ψ controls the degree of time inconsistency: if $\psi = 0$ then the Ramsey outcome is sustainable because there are no benefits of deviating from the optimal plan ex-post. In contrast, if ψ is large then there is a much higher temptation to deviate ex-post.

Suppose now that there are two types of policy makers, each associated with a different value of ψ. The high cost type has $\psi = \psi_H > 0$ and the low cost type has $\psi = \psi_L = 0$. The low cost type then has no incentive to deviate ex-post and thus represents the commitment type in our baseline model. It also corresponds to the "conservative central banker" as in Rogoff (1985) since if private agents know they are facing the low cost type with probability one then the Ramsey outcome can be implemented. To keep the symmetry with the previous analyses, we assume that the social welfare function used by the rule designer to evaluate outcomes is $w(x, \pi; \psi_H)$.

Consider the twice-repeated problem. The characterization in the terminal period does not change relative to the case analyzed previously. Thus, the value for the rule designer is $W_0(\rho)$, where ρ is the prior of facing the low cost type, the value for the high cost type is $V_0(\rho; \psi_H) = V_0(\rho)$, and the value for the low cost type is $V_0(\rho; 0) = w(x_0(\rho), \pi = 0)$.

Consider now the rule designer's problem in the first period. The difference with problem (8) is that we have to add an incentive compatibility constraint for the low cost type (commitment type),

$$w(x, \pi_r; 0) + \beta_o V_0\left(\rho_c'; 0\right) \geqslant w(x, \pi; 0) + \beta_o V_0\left(\rho'(\pi); 0\right) \quad \forall \pi$$

where $\rho'(\pi)$ is the posterior after observing policy π and the low cost type's discount factor is β_o. Since $w_\pi(x, \pi; 0) = 0$ for all (x, π) then we can rewrite the constraint above a $\beta_o V_0(\rho_c'; 0) \geqslant \beta_o V_0(\rho'(\pi); 0)$ or, since $V_0(\rho; 0)$ is strictly increasing in ρ, as

$$\rho_c' = \rho'(\pi_r) \geqslant \rho'(\pi) \quad \forall \pi. \tag{11}$$

The incentive compatibility constraint for the low type (11) is satisfied in the separation regime as $\rho_c' = 1$ so the rule designer can attain the same value. We now turn to analyze whether the pooling regime is feasible. The answer to this question depends on the specification of off-path beliefs. Clearly, it is possible to specify the off-path beliefs as follows

$$\rho'(\pi) = \begin{cases} \rho & \text{if } \pi = \pi_r \\ 0 & \text{if } \pi \neq \pi_r \end{cases} \tag{12}$$

This choice is consistent with Bayes' rule on-path, trivially satisfies (11), and so supports the pooling outcome described above. An unappealing feature of (12) is that implementing tougher policies ex-post reduces the policy maker's reputation. If we restrict specifying beliefs such that $\rho'(\pi)$ is strictly decreasing in π then pooling is not feasible and the separating regime is the only solution for all levels of reputation. The restriction is intuitive as it imposes that if the deviation is relatively more advantageous for the low cost type then the prior is going up.

4 Transparency of Rules

We now study the implications of our theory for the optimal degree of transparency of the rule. Should the rule be designed so that a deviation by the policy maker is easily detectable? In other words, we ask if prefer perfect monitoring is always desirable. Conventional wisdom suggests that for a typical repeated policy game with no reputational considerations, perfect monitoring is always desirable. In contrast, we show that with reputational considerations, perfect monitoring is desirable only for low levels of reputation while imperfect monitoring is desirable for high levels of reputation.

4.1 Optimal Degree of Monitoring

We first consider the case in which the rule designer can control the degree to which private agents and future rule designers can monitor the policies chosen by the policy maker. In particular, suppose private agents cannot directly observe the policy π but they only can observe a signal $\tilde{\pi} = \pi + \varepsilon$ where $\varepsilon \sim N\left(0, \sigma_\varepsilon^2\right)$. The rule designer can choose the standard deviation of the noise, σ_ε, as part of the optimal design of the rule. We interpret a choice of large noise as standing in for complicated rules whose deviations are hard to detect for the private agents. We say that a rule is *transparent* if σ_ε is low and *opaque* is σ_ε is high.

For a given σ_ε, the law of motion for beliefs is

$$\rho'\left(\tilde{\pi}, \rho\right) = \frac{\rho \Pr\left(\tilde{\pi} | \pi_c\right)}{\rho \Pr\left(\tilde{\pi} | \pi_c\right) + (1-\rho) \Pr\left(\tilde{\pi} | \pi_o\right)} = \frac{\rho g\left(\tilde{\pi} - \pi_c\right)}{\rho g\left(\tilde{\pi} - \pi_c\right) + (1-\rho) g\left(\tilde{\pi} - \pi_o\right)} \tag{13}$$

where g is the PDF of a Normal distribution with mean zero and variance σ_ε^2. We can then write the planner's problem for the twice repeated economy as

$$\max_{x, \pi_c, \pi_o, \sigma_\varepsilon} \rho \left[w\left(x, \pi_c\right) + \beta \int W_0\left(\rho'\left(\pi_c + \varepsilon, \rho\right)\right) g(\varepsilon)\, d\varepsilon \right] + (1-\rho)\left[w\left(x, \pi_o\right) + \beta \int W_0\left(\rho'\left(\pi_o + \varepsilon, \rho\right)\right) g(\varepsilon)\, d\varepsilon \right] \tag{14}$$

subject to the implementability condition, $x = \rho\pi_c + (1-\rho)\pi_o$, the incentive compatibility constraint for the optimizing type,

$$w\left(x, \pi_o\right) + \beta_o \int V_0\left(\rho'\left(\pi_o + \varepsilon, \rho\right)\right) g(\varepsilon)\, d\varepsilon \geqslant w(x, \pi) + \beta_o \int V_0\left(\rho'\left(\pi + \varepsilon, \rho\right)\right) g(\varepsilon)\, d\varepsilon \quad \forall \pi, \tag{15}$$

and the law of motion for beliefs (13). Note that W_0 and V_0 are the static values and so are not affected by σ_ε.

The next Proposition establishes that for low levels of reputation it is optimal to have perfectly transparent rules ($\sigma_\varepsilon = 0$) while for higher values of reputation it is optimal to have opaque rules :

Proposition 5. *Under Assumptions 1–4:*

1. *For* ρ *close to zero there is pooling and signals are perfectly informative,* $\sigma_\varepsilon = 0$

2. *For* ρ *close to one there is separation and signals are not perfectly informative,* $\sigma_\varepsilon > 0$

Consider first low levels of reputation. From Proposition 1, we know that if signals are perfectly informative, it is optimal to be in the pooling regime so $\pi_o = \pi_c$. Conditional on pooling, it is preferable to choose $\sigma_\varepsilon = 0$ to relax the incentive constraint (15). In fact, without noise, (15) reduces to

$$w(x, \pi_o) + \beta_o V_0(\rho) \geqslant w(x, \pi) + \beta_o V_0(0) \quad \forall \pi \tag{16}$$

and so the spread in continuation values $[V_0(\rho) - V_0(0)]$ provides the maximal incentives to the optimizing type. To see this, first note that for any $\sigma_\varepsilon > 0$

$$\int V_0\left(\rho'(\pi + \varepsilon, \rho)\right) g(\varepsilon)\, d\varepsilon > V_0(0)$$

so the right side of (15) is the lowest at $\sigma_\varepsilon = 0$. Second, by concavity of V_0 we have that

$$V_0(\rho) > \int V_0\left(\rho'(\pi_o + \varepsilon, \rho)\right) g(\varepsilon)\, d\varepsilon$$

since $\rho = \int \rho'(\pi_o + \varepsilon, \rho)\, g(\varepsilon)\, d\varepsilon$ so the left side of (15) is the highest at $\sigma_\varepsilon = 0$. Thus, since we know that for low levels of reputation pooling is preferable than separating we have that the optimal rule has pooling and it is perfectly transparent.

Consider now high levels of reputation. Suppose by way of contradiction that it is optimal to be in the separating regime ($\pi_o \neq \pi_c$) with perfectly informative signals, $\sigma_\varepsilon = 0$. Since types are perfectly revealed at the end of the first period we have that $\rho' \in \{0, 1\}$ and the only incentive compatible policy for the optimizing type is $\pi_o = \pi^*(x)$. Note that we can support the same policies by choosing $\sigma_\varepsilon = \infty$. This alternative rule has the same static payoff but prevents learning about the regulator's type and therefore $\rho' = \rho$ because the signal $\tilde{\pi}$ is totally uninformative. This increases the expected continuation value because uncertainty is beneficial, $W(\rho) > \rho W(1) + (1 - \rho) W(0)$. Thus, the rule designer's payoff is strictly higher and therefore it cannot be that $\sigma_\varepsilon = 0$. In principle, it may be optimal to choose an intermediate value for the noise σ_ε to induce optimizing type to do something better than the static best response.

4.2 Optimal Tenure

The results in Proposition 5 are also informative about the optimal tenure of the policy maker. In fact, an alternative instrument for the rule designer to separate the static policy choice from the evolution of the reputation of the policy maker in subsequent periods is to terminate the current policy maker after one period. This is equivalent to choosing a perfectly opaque rule with $\sigma_\varepsilon = \infty$. Thus early termination (one period tenure) is optimal when the reputation of a new policy maker is sufficiently high.

Consider the twice repeated environment. To obtain a tighter characerization we prove our result for the Barro-Gordon model. Suppose that in the first period the rule designer can choose a regulation π_c and whether to terminate the policy maker after one period. The prior that a new policy maker is the commitment type is ρ and is constant in both periods. We assume that the termination choice cannot be made contingent on the outcome at the end of the period. It is clear that the rule designer's problem is the same as the one in (14) with the additional restriction that $\sigma_\varepsilon \in \{0, \infty\}$.

Proposition 6. *In the Barro-Gordon model, there exists $\rho^{**} < \rho_1^*$ such that:*

1. *For $\rho \leqslant \rho^{**}$ it is optimal to pool and not terminate the policy maker after one period;*
2. *For $\rho \geqslant \rho^{**}$ it is optimal to separate and terminate the policy maker after one period.*

Consider first case in which pooling has static benefits, $\Delta\omega(\rho) \geqslant 0$. As shown in Figure 2, this is true for $\rho \in [0, \rho^{**}]$ where ρ^{**} is defined as $\Delta\omega(\rho^{**}) = 0$. In this case, the rule designer does not want to terminate the policy maker as it would tighten the incentive compatibility constraint without changing the continuation value. Thus in this region it is optimal to not fire the policy maker.

Consider next the case in which there are static losses of pooling in that $\Delta\omega(\rho) < 0$. This is true for levels of reputation above the cutoff ρ^{**}. For these levels of reputation, if the rule designer keeps the same policy maker in office, the rule designer must trade off the static losses of pooling against the dynamic benefits. However, when the rule designer terminates the policy maker after one period, it can achieve both the static benefits associated with separation and the dynamic benefits associate with pooling. This is because replacing the policy maker after one period prevents learning and does not require the commitment type to implement a lenient policy in order to do so.

4.3 Stochastic Rules

An alternative way of introducing opacity in rules is to allow the rule designer to choose stochastic rules even though fundamentals are deterministic. The rule designer can now choose a rule that consists of a set of policies, Σ_c, and a probability distribution over these polices, σ_c. We can interpret this as introducing clauses that allow policies to be conditioned on irrelevant details. The commitment type will then draw a policy from this distribution. The optimizing type can also randomize across policies. We will denote its strategy as σ_o.

The rule designer's problem is then

$$\max_{x,\sigma_o,\sigma_c} \int \left[w\left(x,\pi\right)+\beta W_0\left(\rho'\left(\pi,\rho\right)\right)\right]\left[\rho\sigma_c\left(\pi\right)+\left(1-\rho\right)\sigma_o\left(\pi\right)\right]d\pi \tag{17}$$

subject to $\sigma_c, \sigma_o \in \Delta\left(\left[\underline{\pi},\bar{\pi}\right]\right)$, the implementability condition,

$$x=\phi\left(\int \pi\left[\rho\sigma_c\left(\pi\right)+\left(1-\rho\right)\sigma_o\left(\pi\right)\right]d\pi\right), \tag{18}$$

the incentive compatibility constraint for the optimizing type, $\forall\pi \in \mathrm{Supp}\sigma_o, \forall\tilde{\pi} \in \mathrm{Supp}\sigma_o \cup \mathrm{Supp}\sigma_c \cup \{\pi^*\left(x\right)\}$

$$w\left(x,\pi\right)+\beta V_0\left(\rho'\left(\pi,\rho\right)\right) \geqslant w\left(x,\tilde{\pi}\right)+\beta V_0\left(\rho'\left(\tilde{\pi},\rho\right)\right), \tag{19}$$

and the evolution of beliefs,

$$\rho'\left(\pi,\rho\right)=\frac{\rho\sigma_c\left(\pi\right)}{\rho\sigma_c\left(\pi\right)+\left(1-\rho\right)\sigma_o\left(\pi\right)}. \tag{20}$$

We say that a rule is stochastic if the support of σ_c contains more than one element while a rule is deterministic if the support of σ_c is a singleton. Similar to Proposition 5, we show that if the policy maker's reputation is high enough then recommending stochastic rules is optimal while if the reputation is sufficiently close to zero then it is optimal to have deterministic rules that provide strong incentives for the optimizing type.

Proposition 7. *Suppose Assumptions 1–4 hold. For ρ close to 1 it is optimal to have stochastic rules. For ρ close to 0 a deterministic rule is optimal and in particular, $\pi_c = \pi_{ico}\left(\rho\right)$ with probability one.*

Consider first the case in which the reputation is close to one. The optimality of stochastic rules follows from properties of Bayes' rule and continuation values being increasing in the prior and does not rely on uncertainty being beneficial. To establish the result, suppose by way of contradiction that it is optimal to choose a rule that recommend policy $\underline{\pi}$ with probability one. This is the best deterministic rule as shown in Proposition 1. Consider a perturbation in which the rule puts a small but positive probability, ε, on the static best response. When ρ is close to one, on observing the static best response, agents attribute this to the perturbation of the commitment type rather than the optimizing type. Consequently the posterior that the policy maker is the commitment type rises sharply which increases the expected continuation value of the perturbation and more than compensates the static losses.[6]

The case with reputation close to zero instead relies on uncertainty being beneficial. The argument here mirrors the one provided to show that randomization by the optimizing type is not optimal. The idea here is that randomization tightens the optimizing type's incentive constraint which results in a more lenient expected policy. This in turn lowers the static payoff in addition to the dynamic losses that arise because uncertainty is beneficial.

The message of this section is that when reputation is low, rules should be transparent and easily interpretable so that deviations are easily detectable. This is because providing incentives to the optimizing type is critical, as in Atkeson et al. (2007). In constrast, when reputation is high, rules should be opaque and hard to interpret. This is because the benefits of maintaining uncertainty about the policy maker's type outweigh the costs associated with looser incentives to the optimizing type. This can account for why countries with low credibility adopt policies like fixed exchanges rates or crawling pegs while countries with high credibility are more likely to have discretionary exchange rate policies.

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[6]Note that forcing the commitment type to randomize reduces the variance of the posterior. In fact, under the deterministic rule with separation, the posterior is 1 with probability ρ and 0 with probability $1-\rho$. Under our perturbation, the posterior is 1 with probability $\rho(1-\varepsilon)$ and $\rho\varepsilon/[\rho\varepsilon+(1-\rho)]$ with probability $\rho\varepsilon+(1-\rho)$.

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Appendix

A Omitted proofs

A.1 Proof of Lemma 1

Suppose first that $\pi_c(\rho) = \underline{\pi}$ for all ρ.. Also assume that $w_x < 0$. The other case follows analogously. Define

$$\bar{w}(x) = w(x, \pi_c)$$

$$w^*(x) = w(x, \pi_o(x))$$

and

$$F(\rho, x(\rho)) = \rho\bar{w}(x(\rho)) + (1-\rho)\, w^*(x(\rho))$$

and so $W_0(\rho) = F(\rho, x(\rho))$. We want to show that $W_0(\rho)$ is concave. We have,

$$\begin{aligned} W_0''(\rho) &= \left[1, x'(\rho)\right] \nabla^2 F(\rho, x(\rho)) \begin{bmatrix} 1 \\ x'(\rho) \end{bmatrix} + \nabla F(\rho, x(\rho)) \begin{bmatrix} 0 \\ x''(\rho) \end{bmatrix} \\ &= 2F_{\rho x} x'(\rho) + F_{xx} x'(\rho)^2 + F_x(\rho, x(\rho))\, x''(\rho) \end{aligned} \tag{21}$$

Thus to prove the result it is sufficient to show that the above expression is negative. We have

$$\begin{aligned} F_{\rho x} &= \bar{w}_x(x(\rho)) - w^*_x(x(\rho)) = w_x(x, \pi_c) - \left[w_x(x, \pi_o(x)) + w_\pi(x, \pi_o(x))\, \pi_o'(x)\right] \\ &= w_x(x, \pi_c) - w_x(x, \pi_o(x)) < 0 \end{aligned}$$

Next, we have

$$x = \phi(\rho\pi_c + (1-\rho)\pi_o)$$

and so

$$x'(\rho) = \phi'[\pi_c - \pi_o] + \phi'(1-\rho)\pi_o' x'$$

$$x'(\rho) = \frac{\phi'[\pi_c - \pi_o]}{[1 - \phi'(1-\rho)\pi_o'(x(\rho))]}$$

which is negative since $\pi_o \geqslant \underline{\pi}$ and $1 - \phi'(1-\rho)\pi_o'(x) \geqslant 0$, where the latter follows from condition 3 of Assumption 2. Next,

$$F_x = \rho w_x(x, \pi_c) + (1-\rho)\, w_x(x, \pi_o) < 0$$

and so

$$F_{xx} = \rho \bar{w}_{xx}(x(\rho)) + (1-\rho) w_x^*(x(\rho))$$
$$= \rho w_{xx}(x, \pi_c) + (1-\rho)\left[w_{xx}(x, \pi_o(x)) + w_{x\pi}(x, \pi)\pi_o'(x)\right]$$

Since

$$w_\pi(x, \pi_o(x)) = 0$$

we have

$$\pi_o'(x) = -\frac{w_{x\pi}(x, \pi_o)}{w_{\pi\pi}(x, \pi_o)}$$

Therefore

$$w_{xx}(x, \pi_o(x)) + w_{x\pi}(x, \pi)\pi_o'(x)$$
$$= \frac{w_{xx}(x, \pi_o(x)) w_{\pi\pi}(x, \pi_o) - w_{x\pi}(x, \pi_o)^2}{w_{\pi\pi}(x, \pi_o)} < 0$$

if w is concave in (x, π). Therefore, $F_{xx} < 0$.

Next, we will show that $x''(\rho) \geqslant 0$.

We have

$$x''(\rho) = \phi''\left[\pi_c - \pi_o + (1-\rho)\pi_o'(\rho)\right]^2 + \phi'\left[-\pi_o'(\rho) + (1-\rho)\pi_o''(\rho) - \pi_o'(\rho)\right]$$
$$= \phi''\left[\pi_c - \pi_o + (1-\rho)\pi_o'(\rho)\right]^2 + \phi'\left[(1-\rho)\pi_o''(\rho) - 2\pi_o'(x)x'(\rho)\right]$$

$$\pi_o''(\rho) = \pi_o''(x)x'(\rho)^2 + \pi_o'(x)x''(\rho)$$

Therefore,

$$x''(\rho) = \frac{\phi''[\pi_c - \pi_o + (1-\rho)\pi_o'(\rho)]^2 + \left[\phi'(1-\rho)\pi_o''(x)x'(\rho)^2 - 2\phi'\pi_o'(x)x'(\rho)\right]}{[1 - \phi'(1-\rho)\pi_o'(x)]}$$

Therefore if $\phi'' \geqslant 0$ and $\pi_o''(x) \geqslant 0$ then $x''(\rho) \geqslant 0$.

Next we show that $\pi_o''(x) \geqslant 0$. We have

$$w_{\pi\pi}(x, \pi_o)\pi_o''(x) + \left[w_{\pi\pi x} + w_{\pi\pi\pi}(x, \pi_o)\pi_o'(x)\right]\pi_o'(x) + w_{x\pi x}(x, \pi_o) + w_{x\pi\pi}(x, \pi_o)\pi_o'(x) = 0$$

$$w_{\pi\pi}(x, \pi_o)\pi_o''(x) + w_{\pi\pi\pi}(x, \pi_o)\pi_o'(x)^2 + 2w_{\pi\pi x}\pi_o'(x) + w_{x\pi x}(x, \pi_o) = 0$$

$$\pi_o''(x) = \frac{w_{\pi\pi\pi}(x, \pi_o)\pi_o'(x)^2 + 2w_{\pi\pi x}\pi_o'(x) + w_{x\pi x}(x, \pi_o)}{(-w_{\pi\pi}(x, \pi_o))}$$

Thus, in order for $\pi_o(x)$ to be convex we need

$$w_{\pi\pi\pi}(x,\pi_o)\,\pi_o'(x)^2 + 2w_{\pi\pi x}\pi_o'(x) + w_{x\pi x}(x,\pi_o) \geqslant 0.$$

Notice that the above is

$$\left[\pi_o'(x),1\right]\nabla^2 w_\pi(x,\pi)\begin{bmatrix}\pi_o'(x)\\ 1\end{bmatrix}$$

which is positive if $w_\pi(x,\pi)$ is convex. Given this lets go back to original expression of interest

$$\begin{aligned}
&2F_{\rho x}x'(\rho) + F_{xx}x'(\rho)^2 + F_x(\rho,x(\rho))\,x''(\rho)\\
<&2F_{\rho x}x'(\rho) + F_x(\rho,x(\rho))\,x''(\rho)\\
=&2\left[w_x(x,\pi_c) - w_x(x,\pi_o(x))\right]x'(\rho) + \left[\rho w_x(x,\pi_c) + (1-\rho)\,w_x(x,\pi_o(x))\right]x''(\rho)
\end{aligned}$$

Lets consider the last expression. We want to show that

$$2\left[w_x(x,\pi_c) - w_x(x,\pi_o(x))\right]x'(\rho) + \left[\rho w_x(x,\pi_c) + (1-\rho)\,w_x(x,\pi_o(x))\right]x''(\rho) \leqslant 0$$

or

$$\frac{2\left[w_x(x,\pi_c) - w_x(x,\pi_o(x))\right]}{\left[\rho w_x(x,\pi_c) + (1-\rho)\,w_x(x,\pi_o(x))\right]} \leqslant -\frac{x''(\rho)}{x'(\rho)}$$

which is true if

$$\frac{2\left[w_x(x,\pi_c) - w_x(x,\pi_o(x))\right]}{\left[\rho\left(w_x(x,\pi_c) - w_x(x,\pi_o(x))\right) + w_x(x,\pi_o(x))\right]} \leqslant \frac{2\pi_o'(x)\,\phi'}{\left[1-\phi'(1-\rho)\,\pi_o'(x(\rho))\right]}$$

or

$$\pi_o'(x)\,\phi' + \frac{w_x(x,\pi_o(x))}{w_x(x,\pi_c)} \geqslant 1$$

Notice that

$$\pi_o'(x)\,\phi' + \frac{w_x(x,\pi_o(x))}{w_x(x,\pi_c)} \geqslant \pi_o'(\underline{x})\,\phi'(\underline{\pi}) + \frac{w_x(\underline{x},\pi_o(\underline{x}))}{w_x(\underline{x},\underline{\pi})} \geqslant 1$$

where we have used the fact that $\pi_o(x), \phi(\pi)$ are convex, $w(x,\pi)$ is concave, and the last inequality follows from condition 3 of Assumption 2.

As a final step we will show that our assumptions imply that π_c is independent of ρ and in particular equals $\underline{\pi}$. The first order condition of the static government's problem wrt π_c is

$$\rho w_\pi(x,\pi) + \left[\rho w_x(x,\pi) + (1-\rho)\,w_x(x,\pi^*(x))\right]\frac{\rho\phi'(\cdot)}{\left[1-\phi'(\cdot)(1-\rho)\,\pi_x^*(x)\right]}$$

Therefore, since by assumption

$$w_{\pi}(x,\pi) + [\rho w_x(x,\pi) + (1-\rho) w_x(x,\pi^*(x))] \frac{\phi'(\cdot)}{[1-\phi'(\cdot)(1-\rho)\pi_x^*(x)]} \leqslant 0$$

it must be that $\pi_c = \underline{\pi}$.

We next show that the two examples satisfy Assumption 2.

Lemma 2. *The Barro-Gordon economy satisfies Assumption 2.*

Proof. Recall that

$$w(x,\pi) - \frac{1}{2}\left[(\psi + x - \pi)^2 + \pi^2\right]$$

Thus,

$$w_{xx} = -1 < 0$$

$$w_{\pi\pi} = -2 < 0$$

and

$$w_{xx}w_{\pi\pi} - w_{x\pi} = 1 > 0$$

Therefore the hessian of $w(x,\pi)$ is negative semi-definite and thus $w(x,\pi)$ is concave. Next, note that $w_{\pi\pi a} = 0, w_{\pi\pi\pi} = 0$ and $w_{\pi aa} = 0$ and so w_{π} is convex.

Next, we have

$$1 > \pi_o'(x)\phi'(\pi) = \frac{1}{2} \geqslant 1 - \frac{w_x(\underline{x},\pi_o(\underline{x}))}{w_x(\underline{x},\underline{\pi})} = \frac{1}{2}$$

and so condition 3 is satisfied.Finally, using a little algebra condition 4 is

$$\left[-2 + \frac{2\rho}{1+\rho}\right]\pi_c$$

which is strictly less than zero for any π_c positive. Thus, it must be $\pi_c = 0$

□

Lemma 3. *If ψ is sufficiently small, then bailout economy satisfies Assumption 2.*

Proof. Recall that

$$w(e,\pi) = -v(e) + p(e)R_H - (1-p(e))(1-\pi)\psi - c(\pi)$$

and thus $\pi_o(e)$ is the solution to

$$(1-p(e))\psi - c'(\pi) = 0$$

Let's first show that $w(e,\pi)$ is concave. We have

$$w_e = -v'(e) + p'(e)\left(R_H + (1-\pi)\psi\right)$$

$$w_{ee} = -v''(e) + p''(e)\left(R_H + (1-\pi)\psi\right) < 0$$

$$w_{e\pi} = -p''(e)\psi > 0$$

$$w_\pi = (1 - p(e))\psi - c'(\pi)$$

$$w_{\pi\pi} = -c''(\pi) \leqslant 0$$

So

$$\begin{aligned} & w_{ee}w_{\pi\pi} - w_{e\pi}^2 \\ = & \left[-v''(e) + p''(e)\left(R_H + (1-\pi)\psi\right)\right]\left(-c''(\pi)\right) - p''(e)^2\psi^2 \end{aligned} \tag{22}$$

The first term is positive since $v'' > 0$, $p'' < 0$, and $c'' > 0$ but $-p''(e)^2\psi^2$ is negative. Clearly, the whole expression is positive if ψ is small enough. Thus the hessian of w is negative semi-definite which implies that w is concave.

Next, we show that $w_\pi(e,\pi)$ is convex. We have

$$w_\pi(e,\pi) = (1-p(e))\psi - c'(\pi)$$

Therefore

$$w_{\pi e} = -p'(e)\psi$$

$$w_{\pi ee} = -p''(e)\psi > 0$$

$$w_{\pi e\pi} = 0$$

$$w_{\pi\pi\pi} = -c'''(\pi) = 0$$

since $c(e)$ is quadratic so $c''' = 0$. Therefore

$$w_{\pi ee}w_{\pi\pi\pi} - w_{\pi e\pi}^2 = \left[-p'(e)\psi\right]\left[-c'''(e)\right] = 0$$

Thus the hessian of w_π is positive semi-definite and so w_π is concave.

Let's now check that condition 3 is satisfied. We have to show that the following two conditions hold

$$1 > \pi_o'(x)\,\phi'(\pi) \tag{23}$$

$$\pi_o'(x)\,\phi'(\pi) \geqslant 1 - \frac{w_x(\underline{x}, \pi_o(\underline{x}))}{w_x(\underline{x}, \overline{\pi})} = 1 - \frac{-v'(\underline{e}) + p'(\underline{e})\,R_H + p'(\underline{e})\,(1-\pi_o(\underline{e}))\,\psi}{-v'(\underline{e}) + p'(\underline{e})\,R_H + p'(\underline{e})\,\psi} = \pi_o(\underline{e}) \tag{24}$$

where

$$\pi_o'(x)\,\phi'(\pi) = \left(-\frac{p'(e)\,\psi}{c''(\pi)}\right)\phi'(\pi)$$

Under our functional form assumption,s

$$p(e) = e^{\alpha}$$

$$c(\pi) = \lambda\pi^2/2$$

$$v(e) = e^2/2,$$

we have that

$$e = \phi(\pi) = \alpha^{\eta}(R_H - \pi)^{\eta} \quad \eta \equiv 1/(2-\alpha) \in (0,1)$$

so

$$\begin{aligned}\left(-\frac{p'(e)\,\psi}{c''(\pi)}\right)\phi'(\pi) &= \frac{-\alpha e^{\alpha-1}\psi}{\lambda}\eta\alpha^{\eta}(R_H-\pi)^{\eta-1} \\ &= \frac{-\alpha\left(\alpha^{\eta}(R_H-\pi)^{\eta}\right)^{\alpha-1}\psi}{\lambda}\left(-\eta\alpha^{\eta}(R_H-\pi)^{\eta-1}\right) \\ &= \frac{\alpha^{1+\eta\alpha}}{2-\alpha}\frac{1}{(R_H-\pi)^{1-\eta\alpha}}\frac{\psi}{\lambda}\end{aligned}$$

and since $\underline{e} = \alpha^{\eta}R_H^{\eta}$ we have that

$$\pi_o(\underline{e}) = \frac{1 - p\left(\alpha^{\eta}R_H^{\eta}\right)}{\lambda}\psi = \left[1 - (\alpha R_H)^{\eta\alpha}\right]\frac{\psi}{\lambda}.$$

By inspection, the first inequality, (23), is satisfied if ψ is sufficiently small while the second inequality, (24), is satisfied if R_H is sufficiently. In fact, as $R_H \to 1/\alpha$, $p(e) \to 1$ so $\pi_o(\underline{e}) \to 0$ while $\left(-\frac{p'(e)\psi}{c''(\pi)}\right)\phi'(\pi) > 0$.

Finally, lets check that condition 4 holds. We have

$$w_\pi (e, \pi) + [\rho w_e (e, \pi) + (1-\rho) w_e (e, \pi^* (e))] \frac{\phi' (\cdot)}{[1 - \phi' (\cdot) (1-\rho) \pi_e^* (e)]}$$
$$= (1 - p(e)) \psi - c'(\pi)$$
$$+ \left[\rho p'(e) [\pi + (1-\pi) \psi] + (1-\rho) [\pi_o (e) + (1 - \pi_o (e)) \psi]\right] \frac{\phi' (\cdot)}{\left[1 - \phi'(\cdot) (1-\rho) \left(-\frac{p'(e)\psi}{c''(\pi)}\right)\right]}$$

which is negative if ψ is sufficiently small since $c' > 0, w_e \geqslant 0$, $\phi' \leqslant 0$, and $1 - \phi' \pi_e^* \geqslant 0$. □

A.2 Optimizing Type Does Not Randomize

We now show that under our assumption it is without loss of generality to consider the case in which the optimizing type either follows the rule with probability one or it chooses its best response deviating from the rule with probability one.

To see this, let's allow the allow the optimizing type to randomize. The value the planner can attain by inducing the optimizing type to follow the rule with probability σ starting with a prior ρ is

$$\begin{aligned} W_{pool} (\sigma, \rho) = {} & [\rho + (1-\rho) \sigma] \left[w (x_{ico} (\sigma, \rho), \pi_{ico} (\sigma, \rho)) + \beta W_0 \left(\rho' (\rho, \sigma)\right)\right] \\ & + (1-\rho)(1-\sigma) [w (x_{ico} (\sigma, \rho), \pi^* (x_{ico} (\sigma, \rho))) + \beta W_0 (0)] \end{aligned}$$

where the evolution of the prior is given by

$$\rho' (\sigma, \rho) = \frac{\rho}{\rho + (1-\rho) \sigma}$$

and $(x_{ico} (\sigma, \rho), \pi_{ico} (\sigma, \rho))$ solves

$$x_{ico} (\sigma, \rho) = \phi ([\rho + (1-\rho) \sigma] \pi_{ico} (\sigma, \rho) + (1-\rho)(1-\sigma) \pi^* (x_{ico} (\sigma, \rho)))$$

$$w (x_{ico} (\sigma, \rho), \pi_{ico} (\sigma, \rho)) + \beta V_0 \left(\rho' (\rho, \sigma)\right) = w (x_{ico} (\sigma, \rho), \pi^* (x_{ico} (\sigma, \rho))) + \beta V_0 (0)$$

that is, they are the most stringent policy that is incentive compatible for the optimizing type given ρ and σ.

The next lemma shows that it is optimal to choose $\sigma = 1$. We will use this intermediate result:

Lemma 4. *Under Assumptions 1 and2,* $V_0 (\rho)$ *is concave.*

Proof. Recall that

$$V_0(\rho) = w(x(\rho), \pi^*(x(\rho)))$$

Then

$$\begin{aligned} V_0'(\rho) &= w_x x'(\rho) + w_\pi \pi^{*\prime} x'(\rho) \\ &= w_x x'(\rho) \end{aligned}$$

and so

$$\begin{aligned} V''(\rho) &= (w_{xx} + w_{x\pi}\pi^*_x)\, x'(\rho)^2 + w_x x''(\rho) \\ &= \left(\frac{w_{xx} w_{\pi\pi}(x, \pi^*) - w_{x\pi}(x, \pi^*)^2}{w_{\pi\pi}(x, \pi_o)} \right) x'(\rho)^2 + w_x x''(\rho) \\ &< 0 \end{aligned}$$

which follows from Assumption 2 and $x''(\rho) \geqslant 0$ where the latter was established in the proof of Lemma 1. □

Lemma 5. *Under Assumptions 1 and2, for all* ρ *and* σ, $W_{pool}(1, \rho) \geqslant W_{pool}(\sigma, \rho)$.

Proof. Consider the deviation that makes the optimizing type to put probability one on the expected policy

$$\pi_{dev} = [\rho + (1-\rho)\sigma]\, \pi_{ico}(\sigma, \rho) + (1-\rho)(1-\sigma)\, \pi^*(x_{ico}(\sigma, \rho))$$

and set $\pi_c = \pi_{dev}$. Clearly, since this deviation leaves the expected π unchanged and so it leaves $x_{ico}(\sigma, \rho)$ unaffected. Because of concavity of w in π and W_0, this policy improves welfare:

$$\begin{aligned} W_{pool}(\sigma, \rho) = {} & [\rho + (1-\rho)\sigma]\, w(x_{ico}(\sigma, \rho), \pi_{ico}(\sigma, \rho)) + (1-\rho)(1-\sigma)\, w(x_{ico}(\sigma, \rho), \pi^*(x_{ico}(\sigma, \rho))) \\ & + [\rho + (1-\rho)\sigma]\, \beta W_0\left(\rho'(\rho, \sigma)\right) + (1-\rho)(1-\sigma)\, \beta W_0(0) \\ & \leqslant w(x_{ico}(\sigma, \rho), [\rho + (1-\rho)\sigma]\, \pi_{ico}(\sigma, \rho) + (1-\rho)(1-\sigma)\, \pi^*(x_{ico}(\sigma, \rho))) + \beta W_0(\rho) \\ & = w(x_{ico}(\sigma, \rho), \pi_{dev}) + \beta W_0(\rho) \end{aligned}$$

We are left to show that this deviation is feasible for the planner in that it satisfies the incentive compatibility constraint for the optimizing type:

$$w(x_{ico}(\sigma, \rho), \pi_{dev}) + \beta V_0(\rho) \geqslant w(x_{ico}(\sigma, \rho), \pi^*(x_{ico}(\sigma, \rho))) + \beta V_0(0)$$

Note that at the original allocation it must be that

$$w\left(x_{ico}\left(\sigma,\rho\right),\pi_{ico}\left(\sigma,\rho\right)\right)+\beta V_{0}\left(\rho'\left(\rho,\sigma\right)\right)\geqslant w\left(x_{ico}\left(\sigma,\rho\right),\pi^{*}\left(x_{ico}\left(\sigma,\rho\right)\right)\right)+\beta V_{0}\left(0\right)$$

$$w\left(x_{ico}\left(\sigma,\rho\right),\pi^{*}\left(x_{ico}\left(\sigma,\rho\right)\right)\right)+\beta V_{0}\left(0\right)\geqslant w\left(x_{ico}\left(\sigma,\rho\right),\pi^{*}\left(x_{ico}\left(\sigma,\rho\right)\right)\right)+\beta V_{0}\left(0\right)$$

multiplying the left and right side of the first line by $[\rho+(1-\rho)\,\sigma]$, the left and right side of the second line by $(1-\rho)\,(1-\sigma)$, and summing up the two resulting equations yields

$$\begin{aligned}w\left(x_{ico}\left(\sigma,\rho\right),\pi^{*}\left(x_{ico}\left(\sigma,\rho\right)\right)\right)+\beta V_{0}\left(0\right)&\leqslant\left[\rho+\left(1-\rho\right)\sigma\right]\left[w\left(x_{ico}\left(\sigma,\rho\right),\pi_{ico}\left(\sigma,\rho\right)\right)+\beta V_{0}\left(\rho'\left(\rho,\sigma\right)\right)\right]\\&\quad+\left(1-\rho\right)\left(1-\sigma\right)\left[w\left(x_{ico}\left(\sigma,\rho\right),\pi^{*}\left(x_{ico}\left(\sigma,\rho\right)\right)\right)+\beta V_{0}\left(0\right)\right]\\&\leqslant w\left(x_{ico}\left(\sigma,\rho\right),\pi_{dev}\right)+\beta V_{0}\left(\rho\right)\end{aligned}$$

where the second inequality follows from concavity of w in π and V_0 in ρ. Thus the proposed deviation is incentive compatible and it increases welfare. Also, since

$$W_{pool}\left(1,\rho\right)=\max_{\pi_c}w\left(\phi\left(\pi_c\right),\pi_c\right)+\beta W_{0}\left(\rho\right)$$

subject to

$$w\left(\phi\left(\pi_c\right),\pi_c\right)+\beta V_{0}\left(\rho\right)\geqslant w\left(\phi\left(\pi_c\right),\pi^{*}\left(\phi\left(\pi_c\right)\right)\right)+\beta V_{0}\left(0\right)$$

so since π_{dev} is feasible for this problem then

$$W_{pool}\left(1,\rho\right)\geqslant w\left(x_{ico}\left(\sigma,\rho\right),\pi_{dev}\right)+\beta W_{0}\left(\rho\right)\geqslant W_{pool}\left(\sigma,\rho\right)$$

as wanted. □

Given this lemma, we can focus on comparing the value of separation with probability one,

$$W_{sep}\left(\rho\right)=W_{0}\left(\rho\right)+\beta\left[\rho W_{0}\left(1\right)+\left(1-\rho\right)W_{0}\left(0\right)\right],$$

and the value of pooling with probability 1,

$$W_{pool}\left(\rho\right)=W_{pool}\left(\sigma,\rho\right).$$

A.3 Proof of Proposition 1

Consider first ρ close to 1. Since the incentive compatibility is binding, we have that for some $\delta>0$, for all ρ

$$W_{ramsey}+\delta\geqslant W_{pool}\left(\rho\right).$$

Clearly, at $\rho = 1$, $W_{sep}(1)$ attains the Ramsey outcome. By continuity, there exists a δ_ε sufficiently small such that for all $\rho \in (1-\varepsilon_\delta, 1)$,

$$W_{sep}(\rho) \geqslant W_{ramsey} + \delta.$$

Combining the two expressions above we have that for all $\rho \in (1-\varepsilon_\delta, 1)$,

$$W_{pool}(\rho) < W_{sep}(\rho)$$

as wanted.

Consider now ρ close to zero and assume that Assumptions 1 and 2 hold. Thus uncertainty is beneficial, $W_0(\rho) > \rho W_0(1) + (1-\rho) W_0(0)$, and the continuation value is higher under pooling than under the separation policy. To show that it is optimal to pool, it is sufficient to show that the static benefits of pooling,

$$\begin{aligned}\Delta\omega(\rho) &= w(\phi(\pi_{ico}(\rho)), \pi_{ico}(\rho)) - W_0(\rho) \\ &= w(\phi(\pi_{ico}(\rho)), \pi_{ico}(\rho)) \\ &\quad - [\rho w(\phi(\rho\pi_c + (1-\rho)\pi_o), \pi_c) + (1-\rho) w(\phi(\rho\pi_c + (1-\rho)\pi_o), \pi_o)]\end{aligned}$$

are positive for priors close to zero. To this end, note that at $\rho = 0$ we have $\Delta\omega(0) = 0$ since $\pi_{ico}(0) = \pi_o(0) = \pi^*(\phi(\pi_o(0)))$. Thus it is sufficient to show that $\Delta\omega'(0) > 0$. Note that

$$\begin{aligned}W_0'(\rho) &= w(\phi(\rho\pi_c + (1-\rho)\pi_o), \pi_c) - w(\phi(\rho\pi_c + (1-\rho)\pi_o), \pi_o) \\ &\quad + [\rho w_x(\phi(\rho\pi_c + (1-\rho)\pi_o), \pi_c) + (1-\rho) w_x(\phi(\rho\pi_c + (1-\rho)\pi_o), \pi_c)] \phi'(\rho\pi_c + (1-\rho)\pi_o) \\ &\quad \times \left[\pi_c - \pi_o + (1-\rho)\pi_o'(\rho)\right]\end{aligned}$$

where we used that $w_\pi(\phi(\rho\pi_c + (1-\rho)\pi_o), \pi_o) = 0$ and

$$w(\phi(\pi_{ico}(\rho)), \pi_{ico}(\rho)) = \left[w_x(\phi(\pi_{ico}(0)), \pi_{ico}(0)) \phi'(\pi_{ico}(0)) + w_\pi(\phi(\pi_{ico}(0)), \pi_{ico}(0))\right] \pi_{ico}'(0).$$

Therefore,

$$\begin{aligned}\Delta\omega'(0) &= \left[w_x(\phi(\pi_{ico}(0)),\pi_{ico}(0))\,\phi'(\pi_{ico}(0)) + w_\pi(\phi(\pi_{ico}(0)),\pi_{ico}(0))\right]\pi'_{ico}(0)\\
&-\{w(\phi(\pi_o(0)),\pi_c) - w(\phi(\pi_o(0)),\pi_o(0))\\
&+w_x(\phi(\pi_o(0)),\pi_o(0))\,\phi'(\pi_o(0))\left[\pi_c-\pi_o+\pi'_o(0)\right]\}\\
&\geqslant \left[w_x(\phi(\pi_{ico}(0)),\pi_{ico}(0))\,\phi'(\pi_{ico}(0)) + w_\pi(\phi(\pi_{ico}(0)),\pi_{ico}(0))\right]\pi'_{ico}(0)\\
&- w_x(\phi(\pi_o(0)),\pi_o(0))\,\phi'(\pi_o(0))\left[\pi_c-\pi_o+\pi'_o(0)\right]\\
&= w_x(\phi(\pi_{ico}(0)),\pi_{ico}(0))\,\phi'(\pi_{ico}(0))\left[\pi'_{ico}(0)-(\pi_c-\pi_o)-\pi'_o(0)\right]\end{aligned}$$

where the first inequality follows from $w(\phi(\pi_o(0)),\pi_c) - w(\phi(\pi_o(0)),\pi_o(0)) < 0$ and the last equality follows from $\pi_{ico}(0) = \pi_o(0)$ which implies that $w_\pi(\phi(\pi_{ico}(0)),\pi_{ico}(0)) = 0$. Since $w_x < 0$ and $\phi' > 0$ then it is sufficient to show that the term in square brackets is negative. To this end, we next show that $\pi'_{ico}(0) = -\infty$ and $-(\pi_c - \pi_o) - \pi'_o(0)$ is bounded.

Let's start with proving that $\lim_{\rho\to 0}\pi'_{ico}(\rho) = -\infty$. Recall that $\pi_{ico}(\rho)$ is implicitly defined by the incentive compatibility constraint

$$w(\phi(\pi_{ico}(\rho)),\pi_{ico}(\rho)) + \beta V_0(\rho) = w(\phi(\pi_{ico}(\rho)),\pi^*(\phi(\pi_{ico}(\rho)))) + \beta V_0(0)\,.$$

Totally differentiating and evaluating at $\rho = 0$ we have that

$$w_\pi(\phi(\pi_{ico}(0)),\pi_{ico}(0))\,\pi'_{ico}(0) = -\beta V'_0(0)$$

where we used that $\pi_{ico}(0) = \pi^*(\phi(\pi_o(0)))$. Since

$$w_\pi(\phi(\pi_{ico}(0)),\pi_{ico}(0)) = w_\pi(\phi(\pi_{ico}(0)),\pi_o(0)) = 0$$

and $-\beta W'_0(0) < 0$, it must be that $\lim_{\rho\to 0}\pi'_{ico}(\rho) = -\infty$ since $w_\pi \geqslant 0$ in the relevant range.

Since $-(\pi_c - \pi_o) \leqslant \bar{\pi} - \underline{\pi}$, we are left to show that $\pi'_o(0)$ is bounded. Recall that $\pi_o(\rho)$ is the solution to

$$w_\pi(\phi(\rho\pi_c + (1-\rho)\pi_o),\pi_o) = 0$$

and so applying the implicit function theorem we have

$$\begin{aligned}\pi'_o(0) &= -\frac{w_{\pi x}(\phi(\pi_o(0)),\pi_o(0))\,x'(0)}{w_{\pi\pi}(\phi(\pi_o(0)),\pi_o(0))}\\
&= -\frac{w_{\pi x}(\phi(\pi_o(0)),\pi_o(0))\,\frac{\phi'[\pi_c-\pi_o]}{[1-\phi'\pi'_o(x)]}}{w_{\pi\pi}(\phi(\pi_o(0)),\pi_o(0))}\end{aligned}$$

Therefore

$$\pi_o'(0) = \pi_x^*(x(0))\,x'(0) = \pi_x^*(x(0))\,\frac{\phi'[\pi_c - \pi_o]}{[1-\phi'\pi_o'(x)]}$$

so

$$|\pi_o'(0)| \leqslant |\pi_x^*(x(0))|\frac{|\phi'(\pi_M)|\,[\bar{\pi} - \underline{\pi}]}{[1-\phi'(\pi_M)\,\pi_x^*(\phi(\pi_M))]} < \infty$$

since ϕ' is assumed to be bounded and $1-\phi'\pi_o'(x)$ is generically not equal to zero. In particular, if the economy satisfies Assumption 2 then $[1-\phi'(\pi_M)\,\pi_x^*(\phi(\pi_M))] > 0$. Suppose not. Then it must be that $[1-\phi'(\pi_M)\,\pi_x^*(\phi(\pi_M))] \leqslant 0$which contradicts Assumption 2 part 3. Thus $\pi_o'(0)$ is bounded.

The above claims imply that $\left[\pi_{ico}'(0) - (\pi_c - \pi_o) - \pi_o'(0)\right] < 0$ so $\Delta\omega'(0) > 0$ as wanted. Q.E.D.

A.4 Proof of Proposition 1 for the Barro-Gordon Example

Consider the Barro-Gordon example with

$$\phi(\rho\pi_c + (1-\rho)\,\pi_0) = \rho\pi_c + (1-\rho)\,\pi_o$$

and the value for the government is

$$w(x,\pi) = -\frac{1}{2}\left[(\psi + x - \pi)^2 + \pi^2\right]$$

so

$$w_\pi = -\left[-(\psi + x) + 2\pi\right]$$

$$w_x = -(\psi + x - \pi)$$

Given this we have

$$\pi^* = \frac{(\psi + x)}{2}$$

Consider first the static problem:

$$W_0(\rho) = \max_{\pi_c,\pi_o,x} -\rho\frac{1}{2}\left[(\psi + x - \pi_c)^2 + \pi_c^2\right] - (1-\rho)\,\frac{1}{2}\left[(\psi + x - \pi_o)^2 + \pi_o^2\right]$$

subject to

$$x = \rho\pi_c + (1-\rho)\,\pi_o$$

$$\pi_o = \frac{(\psi + x)}{2}$$

Combining the two constraints we can express π_o and x in term of π_c as

$$\pi_o = \frac{\psi + \rho\pi_c}{(1+\rho)} = \frac{\rho}{1+\rho}\pi_c + \frac{\psi}{(1+\rho)}$$

and

$$x = \frac{2\rho}{1+\rho}\pi_c + \frac{(1-\rho)}{(1+\rho)}\psi$$

Therefore, substituting into the objective function we obtain

$$\begin{aligned} W_0(\rho) &= \max -\frac{1}{2}\left(\rho\left[\left(\psi + \frac{2\rho}{1+\rho}\pi_c + \frac{(1-\rho)}{(1+\rho)}\psi - \pi_c\right)^2 + \pi_c^2\right]\right. \\ &\quad \left.+ (1-\rho)\left[\left(\psi + \frac{2\rho}{1+\rho}\pi_c + \frac{(1-\rho)}{(1+\rho)}\psi - \frac{\rho}{1+\rho}\pi_c - \frac{\psi}{(1+\rho)}\right)^2 + \left(\frac{\rho}{1+\rho}\pi_c + \frac{\psi}{(1+\rho)}\right)^2\right]\right) \\ &= \max_{\pi_c} -\frac{1}{2}\left(\rho\left[\left(\frac{2}{(1+\rho)}\psi - \left(\frac{1-\rho}{1+\rho}\right)\pi_c\right)^2 + \pi_c^2\right] + (1-\rho)\left[2\left(\frac{\psi}{(1+\rho)} + \frac{\rho}{1+\rho}\pi_c\right)^2\right]\right) \end{aligned}$$

the foc for the above problem is

$$\begin{aligned} 0 &= \left[-\left(\left(\frac{1-\rho}{1+\rho}\right)\frac{4\rho}{(1+\rho)}\psi - \rho\left(\frac{1-\rho}{1+\rho}\right)2\left(\frac{1-\rho}{1+\rho}\right)\pi_c\right) + 2\rho\pi_c\right] \\ &\quad + \left[4\frac{\rho(1-\rho)}{1+\rho}\left(\frac{\psi}{(1+\rho)} + \frac{\rho}{1+\rho}\pi_c\right)\right] \end{aligned}$$

which implies that

$$\pi_c = 0$$

Therefore

$$\pi_o(\rho) = \frac{\psi}{(1+\rho)}$$

$$x(\rho) = \frac{(1-\rho)}{(1+\rho)}\psi$$

and

$$W_0(\rho) = -\frac{\psi^2}{(1+\rho)}$$

Also it is worth noting that

$$W_0''(\rho) = -2\frac{\psi^2}{(1+\rho)} < 0$$

Consider now the two-period problem. Let the value for the optimizing type in the terminal period be

$$V(\rho) = -\frac{1}{2}\left[(\psi + x(\rho) - \pi^*)^2 + (\pi^*)^2\right]$$
$$= -\left(\frac{\psi}{(1+\rho)}\right)^2$$

Lets consider the pooling case first. The optimal rule solves

$$\max_{\pi,x} -\frac{1}{2}\left[(\psi + x - \pi)^2 + \pi^2\right] + \beta W_0(\rho)$$

subject to

$$x = \pi$$

$$-\frac{1}{2}\left[(\psi + x - \pi)^2 + \pi^2\right] + \beta_o V(\rho) \geqslant -\frac{1}{2}\left[(\psi + x - \pi^*)^2 + \pi^{*2}\right] + \beta_o V(0)$$

or

$$\max_{\pi} -\frac{1}{2}\left[\psi^2 + \pi^2\right] + \beta W_0(\rho)$$

subject to

$$-\frac{1}{2}\left[\psi^2 + \pi^2\right] - \beta_o\left(\frac{\psi}{(1+\rho)}\right)^2 = -\frac{(\psi+\pi)^2}{4} - \beta_o\psi^2$$

Thus the solution is pinned down by the last constraint. Solving for π we have

$$\pi_{ico}(\rho) = \psi\left(1 - \sqrt{4\beta_o\left[1 - \left(\frac{1}{(1+\rho)}\right)^2\right]}\right)$$

(Note that $\pi_{ico}(0) = \psi = \pi_o(0)$ and $\pi_{ico}(1) = \psi\left(1 - \sqrt{3\beta_o}\right)$ so it must be that $\beta_o < 1/3$.) So the payoff from pooling is

$$W^{pool}(\rho) = -\psi^2\left[\frac{1}{2}\left[1 + \left(1 - \sqrt{4\beta_o\left[1 - \left(\frac{1}{(1+\rho)}\right)^2\right]}\right)^2\right] + \beta\frac{1}{(1+\rho)}\right]$$
$$= -\psi^2\left[\frac{1}{2}\left[1 + \left(1 - \sqrt{4\beta_o\left[1 - \left(\frac{1}{(1+\rho)}\right)^2\right]}\right)^2\right] + \beta\frac{1}{(1+\rho)}\right]$$

The value of separation is

$$\begin{aligned} W^{sep}(\rho) &= W_0(\rho) + \beta\left[\rho W_0(1) + (1-\rho) W_0(0)\right] \\ &= -\psi^2 \left[\frac{1}{(1+\rho)} + \beta\left[1 - \frac{\rho}{2}\right]\right] \end{aligned}$$

Let's consider

$$\begin{aligned} & W^{pool}(\rho) - W^{sep}(\rho) \\ =& \psi^2 \left(- \left[\frac{1}{2}\left[1 + \left(1 - \sqrt{4\beta_o\left[1 - \left(\frac{1}{(1+\rho)}\right)^2\right]}\right)^2\right] + \beta \frac{1}{(1+\rho)}\right] \right. \\ & \left. + \left[\frac{1}{(1+\rho)} + \beta\left[1 - \frac{\rho}{2}\right]\right]\right) \\ & = \psi^2 \left(\left[\frac{1}{(1+\rho)} - \frac{1}{2}\left[1 + \chi(\rho)\right]\right] + \beta\left[1 - \frac{\rho}{2} - \frac{1}{1+\rho}\right]\right) \end{aligned}$$

where

$$\begin{aligned} z(\rho) &\equiv \left(1 - \sqrt{4\beta_o\left[1 - \left(\frac{1}{(1+\rho)}\right)^2\right]}\right)^2 \\ &= \left(1 - \frac{2\xi}{(1+\rho)}\sqrt{\left[(1+\rho)^2 - 1\right]}\right)^2 \end{aligned}$$

where

$$\xi \equiv \sqrt{\beta_o}$$

At $\rho = 0$ the above is 0 while at $\rho = 1$

$$\psi^2 \left(\left[-\frac{1}{2}\left(1 - \sqrt{4\beta_o\left[1 - \left(\frac{1}{2}\right)^2\right]}\right)^2 \right] \right) < 0$$

Next, let's look at the slope

$$\psi^2 \left(\left[-\frac{1}{(1+\rho)^2} - \frac{1}{2} z'(\rho) \right] + \beta\left[-\frac{1}{2} + \frac{1}{(1+\rho)^2} \right] \right)$$

$$\chi'(\rho) = -\left(8\beta_o \frac{1}{(1+\rho)^3}\right)\left(\left(4\beta_o\left[1 - \left(\frac{1}{(1+\rho)}\right)^2\right]\right)^{-\frac{1}{2}} - 1\right)$$

Notice that as $\rho \to 0$ the above goes to $-\infty$. Therefore the slope at $\rho = 0$ is

$$\psi^2 \left(\left[-1 - \frac{1}{2} z'(\rho) \right] + \frac{\beta}{2} \right) \to \infty$$

so that near 0 there definitely exists a region of pooling.

In general to get pooling we need

$$\beta > \frac{\left[-\frac{1}{(1+\rho)} + \frac{1}{2}\left[1 + z(\rho)\right] \right]}{\left[1 - \frac{\rho}{2} - \frac{1}{1+\rho} \right]}$$

Let

$$F(\rho) = \frac{\left[-\frac{1}{(1+\rho)} + \frac{1}{2}\left[1 + z(\rho)\right] \right]}{\left[1 - \frac{\rho}{2} - \frac{1}{1+\rho} \right]}$$

We have

$$F(0) = \frac{0}{0}$$

So

$$\lim_{\rho \to 0} = \frac{\left[1 + \frac{1}{2} z'(0) \right]}{\frac{1}{2}} = -\infty$$

and

$$F(1) = \frac{\left[-\frac{1}{2} + \frac{1}{2}\left[1 + z(1)\right] \right]}{\left[\frac{1}{2} - \frac{1}{2} \right]} = \infty$$

We next show that $F(\rho)$ is monotone increasing in ρ so that there exists a cutoff ρ^* such that it is optimal to pool for $\rho < \rho^*$ and it is optimal to separate for $\rho > \rho^*$. To this end, note that

$$\begin{aligned} F(\rho) &= \frac{\left[-\frac{1}{(1+\rho)} + \frac{1}{2}\left[1 + z(\rho)\right] \right]}{\left[1 - \frac{\rho}{2} - \frac{1}{1+\rho} \right]} \\ &= \frac{-2 + (1+\rho)\left[1 + z(\rho)\right]}{\rho(1-\rho)} \end{aligned}$$

So

$$F'(\rho) = \frac{\rho(1-\rho)\left[(1+\rho) z'(\rho) + 1 + z(\rho)\right] - \left[-2 + (1+\rho)\left[1 + z(\rho)\right]\right]\left[1 - 2\rho\right]}{\rho^2 (1-\rho)^2}$$

the denominator is positive and so we want to sign the numerator.

Lets do some preliminary calculations. We know

$$\sqrt{z(\rho)} = 1 - 2\xi \frac{\sqrt{\left[(1+\rho)^2 - 1\right]}}{(1+\rho)}$$

and so

$$\frac{2\xi}{(1+\rho)} = \frac{1 - \sqrt{z(\rho)}}{\sqrt{\left[(1+\rho)^2 - 1\right]}}$$

Therefore

$$\begin{aligned}
z'(\rho) &= -\left(8\beta_o \frac{1}{(1+\rho)^3}\right)\left(\left(4\beta_o\left[1 - \left(\frac{1}{(1+\rho)}\right)^2\right]\right)^{-\frac{1}{2}} - 1\right) \\
&= -\left(8\xi^2 \frac{1}{(1+\rho)^3}\right)\left(\frac{1}{2\xi\frac{\sqrt{[(1+\rho)^2-1]}}{(1+\rho)}} - 1\right) \\
&= -\left(8\xi^2 \frac{1}{(1+\rho)^3}\right)\frac{\sqrt{z(\rho)}}{1 - \sqrt{z(\rho)}} \\
&= -\left(\frac{2}{(1+\rho)}\left(\frac{2\xi}{(1+\rho)}\right)^2\right)\frac{\sqrt{z(\rho)}}{1 - \sqrt{z(\rho)}} \\
&= -\left(\frac{2}{(1+\rho)}\frac{\left(1 - \sqrt{z(\rho)}\right)^2}{\left[(1+\rho)^2 - 1\right]}\right)\frac{\sqrt{z(\rho)}}{1 - \sqrt{z(\rho)}} \\
&= -\frac{2}{(1+\rho)}\frac{\left(\sqrt{z(\rho)} - z(\rho)\right)}{\left[(1+\rho)^2 - 1\right]} \\
&= -\frac{2}{(1+\rho)}\frac{\left(\sqrt{z(\rho)} - z(\rho)\right)}{\rho(2+\rho)}
\end{aligned}$$

Next lets consider the numerator of $F'(\rho)$:

$$\rho(1-\rho)\left[(1+\rho)z'(\rho) + 1 + z(\rho)\right] - \left[-2 + (1+\rho)\left[1 + z(\rho)\right]\right]\left[1 - 2\rho\right]$$

The first term is

$$\begin{aligned}
&\rho\left(1-\rho\right)\left[\left(1+\rho\right)z'\left(\rho\right)+1+z\left(\rho\right)\right] \\
&=\left(1-\rho\right)\left[\frac{-2\sqrt{z\left(\rho\right)}+\left(2+\rho\left(2+\rho\right)\right)z\left(\rho\right)+\rho\left(2+\rho\right)}{\left(2+\rho\right)}\right]
\end{aligned}$$

while the second term is

$$\begin{aligned}
&[-2+\left(1+\rho\right)[1+z\left(\rho\right)]][1-2\rho] \\
&=-2[1-2\rho]+\left(1+\rho\right)[1-2\rho]+\left(1+\rho\right)[1-2\rho]z\left(\rho\right)
\end{aligned}$$

Lets combine these two terms

$$\begin{aligned}
&\left(1-\rho\right)\left[\frac{-2\sqrt{z\left(\rho\right)}+\left(2+\rho\left(2+\rho\right)\right)z\left(\rho\right)+\rho\left(2+\rho\right)}{\left(2+\rho\right)}\right] \\
&+2[1-2\rho]-\left(1+\rho\right)[1-2\rho]-\left(1+\rho\right)[1-2\rho]z\left(\rho\right) \\
=&\left[\frac{-2\left(1-\rho\right)\sqrt{z\left(\rho\right)}+[2\left(1-\rho\right)+\rho\left(1-\rho\right)\left(2+\rho\right)-\left(2+\rho\right)\left(1+\rho\right)[1-2\rho]]z\left(\rho\right)+\left(2+\rho\right)\left(1-\rho\right)^2}{\left(2+\rho\right)}\right] \\
=&\left[\frac{-2\left(1-\rho\right)\sqrt{z\left(\rho\right)}+\left[\rho+4\rho^2+\rho^3\right]z\left(\rho\right)+\left(2+\rho\right)\left(1-\rho\right)^2}{\left(2+\rho\right)}\right] \\
>&\left[\frac{-2\left(1-\rho\right)\sqrt{z\left(\rho\right)}+\left[\rho+4\rho^2\right]z\left(\rho\right)+\left(2+\rho\right)\left(1-\rho\right)^2}{\left(2+\rho\right)}\right] \\
=&\left[\frac{\left(1-\rho\right)\left[\left(2+\rho\right)\left(1-\rho\right)-2\sqrt{z\left(\rho\right)}\right]+\left[\rho+4\rho^2\right]z\left(\rho\right)}{\left(2+\rho\right)}\right]
\end{aligned}$$

Now notice that if $\left[\left(2+\rho\right)\left(1-\rho\right)-2\sqrt{z\left(\rho\right)}\right]\geqslant 0$ then we are done. Suppose not, i.e.

$$\left(1-\rho\right)<\frac{2\sqrt{z\left(\rho\right)}}{\left(2+\rho\right)}$$

Therefore

$$
\begin{aligned}
&\left[\frac{(1-\rho)\left[(2+\rho)(1-\rho)-2\sqrt{z(\rho)}\right]+\left[\rho+4\rho^{2}\right]z(\rho)}{(2+\rho)}\right] \\
&> \left[\frac{\frac{2\sqrt{z(\rho)}}{(2+\rho)}\left[(2+\rho)(1-\rho)-2\sqrt{z(\rho)}\right]+\left[\rho+4\rho^{2}\right]z(\rho)}{(2+\rho)}\right] \\
&= \left[\frac{2(1-\rho)\sqrt{z(\rho)}-z(\rho)\left[\frac{4}{(2+\rho)}-\rho-4\rho^{2}\right]}{(2+\rho)}\right]
\end{aligned}
$$

Since $z(\rho) < 1$

$$\sqrt{z(\rho)} > z(\rho)$$

It is also true that $2(1-\rho) > \left[\frac{4}{(2+\rho)}-\rho-4\rho^{2}\right]$. To see this suppose not. Then

$$2(1-\rho)-\left[\frac{4}{(2+\rho)}-\rho-4\rho^{2}\right] < 0$$

or

$$4\rho^{2}(2+\rho)-\rho^{2} < 0$$

which is a contradiction. Therefore, $F'(\rho) > 0$.

A.5 Assumption 4 Satisfied in our Examples

Lemma 6. *In our two examples, the gains from going to best response are decreasing in π. In general, this is true if* $\frac{w_x(\phi(\pi),\pi^*(\phi(\pi)))}{w_x(\phi(\pi),\pi)}$ *is close enough to one.*

Proof. In the Barro-Gordon model:

$$
\begin{aligned}
G(\pi) &= -\frac{1}{2}\left[\left(\psi+\pi-\frac{\psi+\pi}{2}\right)^{2}+\left(\frac{\psi+\pi}{2}\right)^{2}\right]+\frac{1}{2}\left[\psi^{2}+\pi^{2}\right] \\
&= -\left(\frac{\psi+\pi}{2}\right)^{2}+\frac{1}{2}\left[\psi^{2}+\pi^{2}\right] \\
&= -\frac{1}{4}\left[\psi^{2}+\pi^{2}+2\psi\pi\right]+\frac{1}{2}\left[\psi^{2}+\pi^{2}\right] \\
&= \frac{1}{4}\left[\psi^{2}+\pi^{2}\right]-\frac{1}{2}\psi\pi
\end{aligned}
$$

so

$$G'(\pi) = \frac{1}{2}(\pi-\psi) = -\frac{1}{2}(\psi-\pi)$$

which is negative for all $\pi \in [0, \psi]$ i.e. between the Ramsey and the Markov which is the relevant range.

In the bailout example:

$$G(\pi) = -(1 - p(\phi(\pi)))(1 - \pi^*(\phi(\pi)))\psi - c(\pi^*(\phi(\pi))) + (1 - p(\phi(\pi)))(1 - \pi)\psi + c(\pi)$$

Since $(1 - p(\phi(\pi)))\psi = c'(\pi^*(\phi(\pi)))$ we can write

$$\begin{aligned} G'(\pi) &= p'(\phi(\pi))\phi'(\pi)(1 - \pi^*(\phi(\pi)))\psi - p'(\phi(\pi))\phi'(\pi)(1 - \pi)\psi - (1 - p(\phi(\pi)))\psi + c'(\pi) \\ &= p'(\phi(\pi))\phi'(\pi)(\pi - \pi^*(\phi(\pi)))\psi - \left[(1 - p(\phi(\pi)))\psi - c'(\pi)\right] \end{aligned}$$

Recall that

$$\pi^*(e) = \frac{(1 - p(e))\psi}{\lambda}$$

Thus,

$$\begin{aligned} G'(\pi) &= p'(\phi(\pi))\phi'(\pi)\left(\pi - \frac{(1 - p(\phi(\pi)))\psi}{\lambda}\right)\psi - [(1 - p(\phi(\pi)))\psi - \lambda\pi] \\ &= \left[\frac{(1 - p(\phi(\pi)))\psi}{\lambda} - \pi\right]\left[-p'(\phi(\pi))\phi'(\pi) - \frac{\lambda}{\psi}\right]\psi \end{aligned}$$

We are now going to show that the first term in square brackets is positive while the second is negative. Let's start with the first. Since we considering $\pi \in [0, \pi_M]$ it must be that for any ψ, λ, and interior π, the first term is positive since $\pi < \pi^*(\phi(\pi))$. Consider next the second term, $[-p'(\phi(\pi))\phi'(\pi) - \lambda/\psi]$. Note that $-p'(\phi(\pi))\phi'(\pi) > 0$ and it is increasing in π. Moreover, $\pi_M(\psi)$ is increasing in ψ. These two observations imply that we can find a ψ sufficiently small the second term is negative for all $\pi \in [0, \pi_M(\psi)]$. Thus for ψ small enough, we have that $G'(\pi) \leqslant 0$.

In general, we have that

$$[w_x(\phi(\pi), \pi^*(\phi(\pi))) - w_x(\phi(\pi), \pi)]\phi'(\pi) - w_\pi(\phi(\pi), \pi) \leqslant 0$$

or

$$w_x(\phi(\pi), \pi)\left[\frac{w_x(\phi(\pi), \pi^*(\phi(\pi)))}{w_x(\phi(\pi), \pi)} - 1\right]\phi'(\pi) - w_\pi(\phi(\pi), \pi) \leqslant 0$$

which is true if $\frac{w_x(\phi(\pi),\pi^*(\phi(\pi)))}{w_x(\phi(\pi),\pi)}$ is close enough to one. □

A.6 Proof of Proposition 2

The proof is by induction. We will consider several claims.

Claim 1. If $W_{t+1}(\rho) > \rho W_{t+1}(1) + (1 - \rho)W_{t+1}(0)$ then $W_t(\rho) > \rho W_t(1) + (1 - \rho)W_t(0)$.

We consider two cases. First, if there is separation:

$$\begin{aligned}W_t(\rho) &= W_0(\rho) + \beta\rho W_{t+1}(1) + \beta(1-\rho)W_{t+1}(0)\\ &\geqslant [\rho W_0(1) + (1-\rho)W_0(0)] + \beta\rho W_{t+1}(1) + \beta(1-\rho)W_{t+1}(0)\\ &= \rho W_t(1) + (1-\rho)W_t(0)\end{aligned}$$

where the first inequality follows from uncertainty being beneficial in static and the last equality follows from the fact that we have

$$W_t(\rho) = W_{t+1}(\rho) = W_0(\rho) \text{ for } \rho \in \{0,1\}.$$

Second, if there is pooling:

$$\begin{aligned}W_t(\rho) &= w(x, \pi_{ico,t}) + \beta W_{t+1}(\rho)\\ &\geqslant W_0(\rho) + \beta[\rho W_{t+1}(1) + (1-\rho)W_{t+1}(0)]\\ &\geqslant [\rho W_0(1) + (1-\rho)W_0(0)] + \beta[\rho W_{t+1}(1) + (1-\rho)W_{t+1}(0)]\\ &= \rho W_t(1) + (1-\rho)W_t(0)\end{aligned}$$

where the first inequality follows from the assumption that it is optimal to pool, the second inequality follows from uncertainty being beneficial in static, and finally we use that $W_t(\rho) = W_{t+1}(\rho) = W_0(\rho)$ for $\rho \in \{0,1\}$. This completes the proof of Claim 1.

Claim 1 and the fact that W_0 satisfies the property implies that for all t, $W_t(\rho)$ satisfies the above property.

Claim 2. $\pi_{ico,t+1}(\rho) \geqslant \pi_{ico,t}(\rho)$.

For any t and $t+1$, we want to show that for any ρ, $\pi_{ico,t+1} \geqslant \pi_{ico,t}$. Suppose by way of contradiction $\pi_{ico,t+1} < \pi_{ico,t}$. By definition, $\pi_{ico,t}$ and $\pi_{ico,t+1}$ satisfy

$$w(\phi(\pi_{ico,t}), \pi_{ico,t}) + \beta V_{t+1}(\rho) = w(\phi(\pi_{ico,t}), \pi^*(\phi(\pi_{ico,t}))) + \beta V_{t+1}(0)$$

$$w(\phi(\pi_{ico,t+1}), \pi_{ico,t+1}) + \beta V_{t+2}(\rho) = w(\phi(\pi_{ico,t+1}), \pi^*(\phi(\pi_{ico,t+1}))) + \beta V_{t+2}(0)$$

or

$$w(\phi(\pi_{ico,t}), \pi^*(\phi(\pi_{ico,t}))) - w(\phi(\pi_{ico,t}), \pi_{ico,t}) = \beta[V_{t+1}(\rho) - V_{t+1}(0)]$$

$$w(\phi(\pi_{ico,t+1}), \pi^*(\phi(\pi_{ico,t+1}))) - w(\phi(\pi_{ico,t+1}), \pi_{ico,t+1}) = \beta[V_{t+2}(\rho) - V_{t+2}(0)]$$

but

$$\beta[V_{t+1}(\rho) - V_{t+1}(0)] > \beta[V_{t+2}(\rho) - V_{t+2}(0)]$$

$$w\left(\phi\left(\pi_{ico,t+1}\right),\pi^*\left(\phi\left(\pi_{ico,t+1}\right)\right)\right)-w\left(\phi\left(\pi_{ico,t+1}\right),\pi_{ico,t+1}\right)$$
$$>w\left(\phi\left(\pi_{ico,t}\right),\pi^*\left(\phi\left(\pi_{ico,t}\right)\right)\right)-w\left(\phi\left(\pi_{ico,t}\right),\pi_{ico,t}\right)$$

since under the contradiction hypothesis

$$\phi\left(\pi_{ico,t}\right)>\phi\left(\pi_{ico,t+1}\right)$$

and the gains from going to best response are decreasing in x, so we have x contradiction. This concludes the proof of Claim 2.

Note that for $\rho=0$, no matter T, we have the Markov outcome. This is because for all t, T, $\beta\left[V_{t+1}(\rho)-V_{t+1}(0)\right]=0$ so $\pi_t(0)=\pi_0(0)$.

Now let

$$\Gamma_t^T=\left\{\rho: W_{pool,t}^T(\rho)\geqslant W_{sep,t}^T(\rho)\right\}$$

be the set of ρ's where it is optimal to pool in period t.

Claim 3. The set of priors where it is optimal to pool is larger if the horizon is longer. That is, $\Gamma_{t+1}^T\subseteq\Gamma_t^T$.

Consider ρ such that at $t+1$ it is optimal to pool so

$$w\left(\phi\left(\pi_{ico,t+1}\right),\pi_{ico,t+1}\right)+\beta W_{t+2}(\rho)\geqslant W_0(\rho)+\beta\left[\rho W_{t+2}(1)+(1-\rho)W_{t+2}(0)\right]$$

or

$$W_0(\rho)-w\left(\phi\left(\pi_{ico,t+1}\right),\pi_{ico,t+1}\right)\leqslant\beta W_{t+2}(\rho)-\beta\left[\rho W_{t+2}(1)+(1-\rho)W_{t+2}(0)\right]$$

Now suppose by way of contradiction that for such ρ at t it is optimal to separate so

$$w\left(\phi\left(\pi_{ico,t}\right),\pi_{ico,t}\right)+\beta W_{t+1}(\rho)<W_0(\rho)+\beta\left[\rho W_{t+1}(1)+(1-\rho)W_{t+1}(0)\right]$$

or

$$W_0(\rho)-w\left(\phi\left(\pi_{ico,t}\right),\pi_{ico,t}\right)>\beta W_{t+1}(\rho)-\beta\left[\rho W_{t+1}(1)+(1-\rho)W_{t+1}(0)\right]$$

Subtracting the expression for $t+1$ from the expression at t we obtain

$$0>w\left(\phi\left(\pi_{ico,t+1}\right),\pi_{ico,t+1}\right)-w\left(\phi\left(\pi_{ico,t}\right),\pi_{ico,t}\right)>\beta\left\{W_{t+1}(\rho)-\left[\rho W_{t+1}(1)+(1-\rho)W_{t+1}(0)\right]\right\}$$
$$-\beta\left\{W_{t+2}(\rho)-\left[\rho W_{t+2}(1)+(1-\rho)W_{t+2}(0)\right]\right\}$$

Thus to get a contradiction it is enough to show that

$$\Delta\Omega_{t+1}(\rho) = \{W_{t+1}(\rho) - [\rho W_{t+1}(1) + (1-\rho) W_{t+1}(0)]\}$$
$$> \Delta\Omega_{t+2}(\rho) = \{W_{t+2}(\rho) - [\rho W_{t+2}(1) + (1-\rho) W_{t+2}(0)]\}$$

which is

$$\Delta\Omega_{t+1}(\rho) = \sum_{j=t+1}^{T} \beta^{j-(t+1)} \left[w\left(\phi\left(\pi_{ico,j}\right), \pi_{ico,j}\right) - (\rho W_0(1) + (1-\rho) W_0(0))\right]$$

$$\Delta\Omega_{t+2}(\rho) = \sum_{j=t+2}^{T} \beta^{j-(t+1)} \left[w\left(\phi\left(\pi_{ico,j}\right), \pi_{ico,j}\right) - (\rho W_0(1) + (1-\rho) W_0(0))\right]$$

so

$$\Delta\Omega_{t+1}(\rho) = [w(\phi(\pi_{ico,t+1}), \pi_{ico,t+1}) - (\rho W_0(1) + (1-\rho) W_0(0))] + \beta\Delta\Omega_{t+2}(\rho)$$

Thus

$$\begin{aligned}
&\Delta\Omega_{t+1}(\rho) - \Delta\Omega_{t+2}(\rho)\\
&= [w(\phi(\pi_{ico,t+1}), \pi_{ico,t+1}) - (\rho W_0(1) + (1-\rho) W_0(0))] - (1-\beta)\Delta\Omega_{t+2}(\rho)\\
&\geqslant [w(\phi(\pi_{ico,t+1}), \pi_{ico,t+1}) - (\rho W_0(1) + (1-\rho) W_0(0))]\\
&- \frac{(1-\beta)\left(1-\beta^{T-(t+2)+1}\right)}{(1-\beta)} [w(\phi(\pi_{ico,t+2}), \pi_{ico,t+2}) - (\rho W_0(1) + (1-\rho) W_0(0))]\\
&= [w(\phi(\pi_{ico,t+1}), \pi_{ico,t+1}) - (\rho W_0(1) + (1-\rho) W_0(0))]\\
&- \left(1-\beta^{T-(t+2)+1}\right) [w(\phi(\pi_{ico,t+2}), \pi_{ico,t+2}) - (\rho W_0(1) + (1-\rho) W_0(0))]\\
&> [w(\phi(\pi_{ico,t+1}), \pi_{ico,t+1}) - (\rho W_0(1) + (1-\rho) W_0(0))]\\
&- [w(\phi(\pi_{ico,t+2}), \pi_{ico,t+2}) - (\rho W_0(1) + (1-\rho) W_0(0))]\\
&= w(\phi(\pi_{ico,t+1}), \pi_{ico,t+1}) - w(\phi(\pi_{ico,t+2}), \pi_{ico,t+2})\\
&> 0
\end{aligned}$$

where the first inequality follows from the fact that the gains from uncertainty (static) are decreasing over time,

$$\{[w(\phi(\pi_{ico,t}), \pi_{ico,t}) - (\rho W_0(1) + (1-\rho) W_0(0))]\}_{t=0}^{T} \downarrow,$$

the second inequality from $\left(1-\beta^{T-(t+2)+1}\right) \in (0,1)$, the last follows from Claim 2. Thus we have a contradiction and so if it is optimal to pool at $t+1$ for ρ then it is optimal to pool at t. This concludes the proof for Claim 3.

Let's now consider the limit at $T \to \infty$. We are particularly interested in the limit of $\left\{\pi^{T}_{ico,0}(\rho)\right\}_{T=1}^{\infty}$. Note also that $\pi^{T}_{ico,t}(\rho) = \pi^{T+1}_{ico,t+1}(\rho)$. From Claim 2, we know that

$$\pi^{T}_{ico,t}(\rho) = \pi^{T+1}_{ico,t+1}(\rho) \geqslant \pi^{T+1}_{ico,t}(\rho)$$

Hence $\left\{\pi^{T}_{ico,0}(\rho)\right\}_{T=0}^{\infty}$ is x decreasing and bounded sequence and so it must converge. Let's denote such limit by $\pi_{ico}(\rho)$. Since from Claim 3 we know that for all T and $\rho \leqslant \rho_1^*$ it is always optimal to pool (other than in the terminal period T) we have that

$$\begin{aligned}
&\sum_{t=0}^{T-1} \beta^{t} w\left(\phi\left(\pi^{T}_{ico,t}(\rho)\right), \pi^{T}_{ico,t}(\rho)\right) + \beta^{T} V_0(\rho) \\
&= w\left(\phi\left(\pi^{T}_{ico,t}(\rho)\right), \pi^{*}\left(\phi\left(\pi^{T}_{ico,t}(\rho)\right)\right)\right) + \beta\frac{1-\beta^{T+1}}{1-\beta} W_0(0)
\end{aligned}$$

Now, since $\pi^{T+1}_{ico,t+1}(\rho) \geqslant \pi^{T+1}_{ico,t}(\rho)$ then

$$\lim_{T\to\infty}\lim_{k\to T} \pi^{T}_{ico,T-k}(\rho) = \pi_{ico}(\rho)$$

and so in the limit we have that

$$w(\phi(\pi_{ico}(\rho)), \pi^{*}(\phi(\pi_{ico}(\rho)))) - w(\phi(\pi_{ico}(\rho)), \pi_{ico}(\rho)) = \frac{\beta}{1-\beta}\left[w(\phi(\pi_{ico}(\rho)), \pi_{ico}(\rho)) - W_0(0)\right]$$

Take two $\rho_1, \rho_2 \leqslant \rho^*$. Then as $T \to \infty$, let $\pi_{ico}(\rho)$ be the limit for $\pi_{ico,0}(\rho_1)$ as $T \to \infty$. At the limit, it must be that

$$\begin{aligned}
&w(\phi(\pi_{ico}(\rho_1)), \pi^{*}(\phi(\pi_{ico}(\rho_1)))) - w(\phi(\pi_{ico}(\rho_1)), \pi_{ico}(\rho_1)) \\
&= \frac{\beta}{1-\beta}\left[w(\phi(\pi_{ico}(\rho_1)), \pi_{ico}(\rho_1)) - W_0(0)\right]
\end{aligned}$$

$$\begin{aligned}
&w(\phi(\pi_{ico}(\rho_2)), \pi^{*}(\phi(\pi_{ico}(\rho_2)))) - w(\phi(\pi_{ico}(\rho_2)), \pi_{ico}(\rho_2)) \\
&= \frac{\beta}{1-\beta}\left[w(\phi(\pi_{ico}(\rho_2)), \pi_{ico}(\rho_2)) - W_0(0)\right]
\end{aligned}$$

Since the equation above has only one solution (other than the Markov equilibrium outcome) in our Barro-Gordon then

$$\pi_{ico}(\rho_1) = \pi_{ico}(\rho_2) = \pi_{ico}.$$

We now turn to show that in the limit there exist x $\rho^* \geqslant \rho_1^*$ such that it is optimal to pool for priors lower than ρ^* and it is optimal to separate for priors larger than ρ^*. Note that

$$\lim_{T\to\infty} w\left(\phi\left(\pi^T_{ico,0}(\rho)\right), \pi^T_{ico,0}(\rho)\right) + \beta W_1^T(\rho) \geqslant \lim_{T\to\infty} \sum_{t=0}^{T-1} \beta^t w\left(\phi\left(\pi^T_{ico,t}(\rho)\right), \pi^T_{ico,t}(\rho)\right)$$
$$= \frac{w(\phi(\pi_{ico}), \pi_{ico})}{1-\beta}$$

so if

$$\frac{w(\phi(\pi_{ico}), \pi_{ico})}{1-\beta} \geqslant W_0(\rho) + \beta\left[\rho\frac{W_0(1)}{1-\beta} + (1-\rho)\frac{W_0(0)}{1-\beta}\right]$$

then it is optimal to pool. If instead

$$\frac{w(\phi(\pi_{ico}), \pi_{ico})}{1-\beta} < W_{sep}(\rho) = W_0(\rho) + \beta\left[\rho\frac{W_0(1)}{1-\beta} + (1-\rho)\frac{W_0(0)}{1-\beta}\right]$$

then it is optimal to separate. To see this, suppose by way of contradiction it is optimal to pool until some $\hat{T} \leqslant T$ so

$$\begin{aligned}
W_{sep}(\rho) &\leqslant w(\phi(\pi_{ico}), \pi_{ico}) + \beta \lim_{T\to\infty} W_1^T(\rho) \\
&= \frac{1-\beta^{\hat{T}}}{1-\beta} w(\phi(\pi_{ico}), \pi_{ico}) + \beta^{\hat{T}} W_{sep}(\rho) \\
&< \left(1-\beta^{\hat{T}}\right) W_{sep}(\rho) + \beta^{\hat{T}} W_{sep}(\rho) = W_{sep}(\rho)
\end{aligned}$$

so we have a contradiction.

To see that there is a cutoff property just note that

$$W_{pool} = \frac{w(\phi(\pi_{ico}), \pi_{ico})}{1-\beta}$$

is constant over $(0, 1]$ while the value of separating $W_{sep}(\rho)$ is strictly increasing in ρ so ρ^* is the unique solution to $W_{pool} = W_{sep}(\rho^*)$. Q.E.D.

A.7 Proof of Proposition 3

Proof. Let's denote the value of the best PBE with prior ρ by $\bar{W}(\rho)$. Consider first the value when $\rho = 0$. Here trigger strategies alone can attains the value of W_{pool} described above:

$$\bar{W}(0) = \max_{\pi} w(\phi(\pi), \pi) + \beta\bar{W}(0)$$

subject to

$$w\left(\phi\left(\pi\right),\pi\right) \geqslant \left(1-\beta\right) w\left(\phi\left(\pi\right),\pi^*\left(\phi\left(\pi\right)\right)\right) + \beta \underline{V}\left(0\right)$$

where the worst equilibrium for the optimizing type is $\underline{V}(0) = W_0(0) / (1-\beta)$. Clearly $\bar{W}(0) = W_{pool}$ and the optimal policy is π_{ico}.

Consider now any $\rho > 0$. If it is optimal to pool at zero it is optimal to pool in any subsequent periods so

$$\bar{W}_{pool}\left(\rho\right) = \frac{w\left(\phi\left(\pi_{ico}\right),\pi_{ico}\right)}{1-\beta} = \bar{W}\left(0\right) = W_{pool}.$$

If instead it is optimal to separate in period 0, the problem solves

$$\bar{W}_{sep}\left(\rho\right) = \max_{\pi_c,\pi_o,x} \rho\left[w\left(x,\pi_c\right) + \beta W_{ramsey}\right] + \left(1-\rho\right)\left[w\left(x,\pi_o\right) + \beta \bar{W}\left(0\right)\right]$$

subject to

$$x = \phi\left(\rho\pi_c + \left(1-\rho\right)\pi_o\right)$$

$$w\left(x,\pi_o\right) + \beta\bar{V}\left(0\right) \geqslant w\left(x,\pi^*\left(x\right)\right) + \beta\underline{V}\left(0\right)$$

Clearly, i) $\bar{W}_{sep}(\rho) > W_{sep}(\rho)$ for all $\rho < 1$ where

$$\begin{aligned} W_{sep}\left(\rho\right) &= W_0\left(\rho\right) + \frac{\beta}{1-\beta}\left[\rho W_0\left(1\right) + \left(1-\rho\right)W_0\left(0\right)\right] \\ &= W_0\left(\rho\right) + \beta\left[\rho W_{ramsey} + \left(1-\rho\right)\underline{W}\left(0\right)\right] \end{aligned}$$

and ii) $\bar{W}_{sep}(0) = \bar{W}(0) = W_{pool}$ and it is strictly increasing in ρ. Thus, for all $\rho > 0$,

$$\bar{W}_{sep}\left(\rho\right) > W_{pool}.$$

To see the last step, note that $W_{ramsey} > \bar{W}(0) = W_{pool}$ so

$$\bar{W}_{sep}\left(\rho\right) > W_{pool} = \max_{\pi_c,\pi_o,x} \rho w\left(x,\pi_c\right) + \left(1-\rho\right)w\left(x,\pi_o\right) + \beta\bar{W}\left(0\right)$$

subject to

$$x = \phi\left(\rho\pi_c + \left(1-\rho\right)\pi_o\right)$$

$$w\left(x,\pi_o\right) + \beta\bar{V}\left(0\right) \geqslant w\left(x,\pi^*\left(x\right)\right) + \beta\underline{V}\left(0\right)$$

and the additional restriction

$$\pi_o = \pi_c.$$

Thus it is always optimal to separate and the limit of the finite horizon converges to the best PBE only for the trivial case $\rho = 1$. □

A.8 Proof of Proposition 4

Proof. Notice that the statically optimal rule chosen by the commitment type is $\underline{\pi}$. To see why note that the first order condition for the commitment type is

$$
\begin{aligned}
& w_{\pi}(x,\pi) + w_{x}(x,\pi)\frac{\phi'(\cdot)}{[1-\phi'(\cdot)(1-\rho)\pi^*_x(x)]} \\
\leqslant & w_{\pi}(x,\pi) + [\rho w_x(x,\pi) + (1-\rho) w_x(x,\pi^*(x))]\frac{\phi'(\cdot)}{[1-\phi'(\cdot)(1-\rho)\pi^*_x(x)]} \\
\leqslant & 0
\end{aligned}
$$

where the first inequality follows from the assumption that $w_{x\pi} \geqslant 0$ and the last inequality from Assumption 2. Let

$$V^c_0(\rho) = w(x_0(\rho), \underline{\pi}),$$

be the value for the commitment type in the terminal period given the prior ρ where

$$x_0(\rho) = \phi(\rho\underline{\pi} + (1-\rho)\pi^*(x_0(\rho))).$$

We can write the value for the commitment type if it chooses to separate as

$$V^c_{sep}(\rho) = w(x_0(\rho), \underline{\pi}) + \beta V^c_0(1).$$

while the value of pooling is

$$V^c_{pool}(\rho) = w(\phi(\pi_{ico}(\rho)), \pi_{ico}(\rho)) + \beta V^c_0(\rho)$$

where π_{ico} solves

$$w(\phi(\pi_{ico}(\rho)), \pi_{ico}(\rho)) + \beta V_0(\rho) = w(\phi(\pi_{ico}(\rho)), \pi^*(\phi(\pi_{ico}(\rho)))) + \beta V_0(0).$$

Therefore,

$$
\begin{aligned}
& V^c_{sep}(\rho) - V^c_{pool}(\rho) \\
= & [w(x_0(\rho), \underline{\pi}) - w(\phi(\pi_{ico}), \pi_{ico})] + \beta[V^c_0(1) - V^c_0(\rho)]
\end{aligned}
$$

First note that for ρ sufficiently large separating has both dynamic gains, $V^c_0(1) - V^c_0(\rho) > 0$, and static gains as $[w(x_0(\rho), \underline{\pi}) - w(\phi(\pi_{ico}), \pi_{ico})] > 0$. In particular, for $\rho \to 1$ we have that the static gains of separating converge to

$$[w(\phi(\underline{\pi}), \underline{\pi}) - w(\phi(\pi_{ico}(1)), \pi_{ico}(1))]$$

which is positive since under our assumption that the Ramsey outcome is not sustainable, $\pi_{ico}(\rho) < \underline{\pi}$. Consequently for ρ large enough the commitment type will choose a tough rule and thus there will be separation.

Next, given some ρ, the commitment type would like to separate if

$$\beta \geqslant \underline{\beta}(\rho) \equiv \frac{[w(\phi(\pi_{ico}(\rho)), \pi_{ico}(\rho)) - w(x_0(\rho), \underline{\pi})]}{\left[V_0^c(1) - V_0^c(\rho)\right]}$$

To show that it is optimal for the optimizing type to separate at $\underline{\pi}$ it must be that

$$w(x_0(\rho), \pi^*(x_0(\rho))) + \beta V_0(0) > w(x_0(\rho), \underline{\pi}) + \beta V_0(1)$$

(note that if the optimizing type mimics the commitment type the posterior jumps to one because we are constructing an equilibrium with separation) or

$$\beta < \bar{\beta}(\rho) \equiv \frac{[w(x_0(\rho), \pi^*(x_0(\rho))) - w(x_0(\rho), \underline{\pi})]}{V_0(1) - V_0(0)}$$

Therefore the equilibrium outcome of the signaling game has separation if

$$\bar{\beta}(\rho) > \beta > \underline{\beta}(\rho)$$

Thus we need to show that such interval exists. For $\rho \to 0$ we have

$$\lim_{\rho \to 0} \bar{\beta}(\rho) = \frac{w(x_0(0), \pi^*(x_0(0))) - w(x_0(0), \underline{\pi})}{V_0(1) - V_0(0)}$$
$$\lim_{\rho \to 0} \bar{\beta}(\rho) = \frac{w(x_0(0), \pi^*(x_0(0))) - w(x_0(0), \underline{\pi})}{V_0^c(1) - V_0^c(0)}$$

since $\pi_{ico}(\rho) \to \pi^*(x_0(0))$. Thus to compare $\bar{\beta}(0)$ and $\underline{\beta}(0)$ we only need to compare the denominators since the numerators are common. In particular, $\bar{\beta}(0) > \underline{\beta}(0)$ if and only if $V_0(1) - V_0(0) < V_0^c(1) - V_0^c(0)$, or

$$w(x_0(1), \pi^*(x_0(1))) - w(x_0(0), \pi^*(x_0(0))) < w(x_0(1), \underline{\pi}) - w(x_0(0), \underline{\pi})$$

or

$$w(x_0(1), \pi^*(x_0(1))) - w(x_0(1), \underline{\pi}) < w(x_0(0), \pi^*(x_0(0))) - w(x_0(0), \underline{\pi})$$

which is satisfied in our two examples. (Note that this property does not contradict Assumption 5 since $\underline{\pi} \neq \phi^{-1}(x_0(0))$.)

In general, we want to show that

$$w(x_0(0), \pi^*(x_0(0))) - w(x_0(0), \underline{\pi}) > w(x_0(1), \pi^*(x_0(1))) - w(x_0(1), \underline{\pi})$$

note that

$$w(x_0(0), \pi^*(x_0(0))) - w(x_0(0), \underline{\pi}) \geqslant w(x_0(0), \pi^*(x_0(1))) - w(x_0(0), \underline{\pi})$$

so we are left to show that

$$w(x_0(0), \pi^*(x_0(1))) - w(x_0(0), \underline{\pi}) > w(x_0(1), \pi^*(x_0(1))) - w(x_0(1), \underline{\pi})$$

Under assumption 1 part a, for $x_H > x_L$

$$\int_{[\underline{\pi},\pi^*]} w_\pi(x_H, \pi)\, d\pi > \int_{[\underline{\pi},\pi^*]} w_\pi(x_L, \pi)\, d\pi$$

and $x_0(0) > x_0(1)$ thus the inequality is satisfied. A similar argument can be made for assumption 1 part 2. □

A.9 Proof of Proposition 7

We start by proving the first result. Consider a ρ such that with a deterministic rule it is optimal to separate so $\pi = \pi_0(\rho)$. The value of this policy is

$$\begin{aligned} W &= [\rho w(x_0, \pi_0) + (1-\rho) w(x_0, \pi^*(x_0))] + \beta [\rho W_0(1) + (1-\rho) W_0(0)] \\ &= W_0(\rho) + \beta [\rho W_0(1) + (1-\rho) W_0(0)] \end{aligned}$$

We now show that if ρ is close to one then this policy is dominated by one that calls for the commitment type to play the static best response with some positive probability. Consider a deviation indexed by $\varepsilon > 0$ sufficiently small so that

$$\pi_c = \begin{cases} \pi_0(\rho) & \text{with pr } 1-\varepsilon \\ 1 & \text{with pr } \varepsilon \end{cases}$$

so after observing a bailout

$$\rho' = \frac{\rho\varepsilon}{\rho\varepsilon + (1-\rho)} = \frac{\rho}{\rho + (1-\rho)/\varepsilon} > 0$$

and after no-bailout $\rho' = 1$. The value of this deviation is then

$$W^{dev}(\varepsilon) = [\rho(1-\varepsilon) w(x_0(\varepsilon), \pi_0) + [\rho\varepsilon + (1-\rho)] w(x_0(\varepsilon), \pi^*(x_0))] + \beta\left[\rho(1-\varepsilon) W_0(1) + [\rho\varepsilon + (1-\rho)] W_0(\rho')\right]$$

Since $W = W^{dev}(0)$ we have

$$W^{dev}(\varepsilon) - W^{dev}(0) = \Delta\omega(\varepsilon) + \beta\Delta\Omega(\varepsilon) \approx \left[\Delta\omega'(\varepsilon) + \beta\Delta\Omega'(\varepsilon)\right]\varepsilon$$

where

$$\Delta\omega(\varepsilon) = [\rho(1-\varepsilon) w(x_0(\varepsilon), \pi_0) + [\rho\varepsilon + (1-\rho)] w(x_0(\varepsilon), \pi^*(x_0))]$$

$$\Delta\Omega(\varepsilon) = \left[\rho(1-\varepsilon) W_0(1) + [\rho\varepsilon + (1-\rho)] W_0(\rho')\right] - [\rho W_0(1) + (1-\rho) W_0(0)]$$

Note that

$$\Delta\Omega'(\varepsilon) = -\rho W_0(1) + \rho W_0(\rho'(\varepsilon)) + [\rho\varepsilon + (1-\rho)] W_0'(\rho'(\varepsilon)) \frac{\partial\rho'}{\partial\varepsilon} = -\rho\left[W_0(1) - W_0(\rho'(\varepsilon))\right] + [\rho\varepsilon + (1-\rho)] W_0'(\rho'(\varepsilon)) \frac{\partial\rho'}{\partial\varepsilon}$$

As $\varepsilon \to 0$

$$\Delta\Omega'(\varepsilon) = -\rho[W_0(1) - W_0(0)] + [(1-\rho)] W_0'(0) \frac{\partial\rho'}{\partial\varepsilon}$$

$$\frac{\partial\rho'}{\partial\varepsilon} = \frac{\rho[\rho\varepsilon + (1-\rho)] - \rho\varepsilon\rho}{[\rho\varepsilon + (1-\rho)]^2} \to \frac{\rho(1-\rho)}{(1-\rho)^2} = \frac{\rho}{(1-\rho)}$$

so for ρ close to one

$$\lim_{\rho\to1}\lim_{\varepsilon\to0} \Delta\Omega'(\varepsilon) \to \infty$$

Thus to show that the deviation is profitable it is sufficient to show that $\Delta\omega > -M$ for some M sufficiently large. Consider

$$\Delta\omega'(\varepsilon) = \rho[w(x_0(\varepsilon), \pi^*(x_0(\varepsilon))) - w(x_0(\varepsilon), \pi_0)] + \{\rho(1-\varepsilon) w_x(x_0(\varepsilon), \pi_0) + [\rho\varepsilon + (1-\rho)] w_x(x_0(\varepsilon), \pi^*(x_0))\} \frac{\partial x_0(\varepsilon)}{\partial\varepsilon} + \rho(1-\varepsilon) w_\pi(x_0(\varepsilon), \pi_0)$$

where

$$\frac{\partial x_0(\varepsilon)}{\partial\varepsilon} = \phi_\pi\rho(\pi^* - \pi_0)$$

Since the first term in square brackets is positive we have that

$$\Delta\omega'(\varepsilon) > \{\rho(1-\varepsilon) w_x(x_0(\varepsilon), \pi_0) + [\rho\varepsilon + (1-\rho)] w_x(x_0(\varepsilon), \pi^*(x_0))\} \frac{\partial x_0(\varepsilon)}{\partial\varepsilon} + \rho(1-\varepsilon) w_\pi(x_0(\varepsilon), \pi_0)$$
$$= \{\rho w_x(x_0, \pi_0) + (1-\rho) w_x(x_0, \pi^*(x_0))\} \phi_\pi \rho(\pi^* - \pi_0) + \rho w_\pi(x_0, \pi_0)$$

with w_x and ϕ_π bounded, as $\rho \to 1$ we have that the last expression converges to

$$w_x(x_0, \pi_0) \phi_\pi(\pi^* - \pi_0) + w_\pi(x_0, \pi_0) > w_x(x_0, \pi_0) \phi_\pi(\pi^* - \pi_0) > w_x(x_0, \pi_0) \phi_\pi(\bar{\pi} - \underline{\pi}) > -M$$

for some $M < \infty$.

(Notice that to derive this result we are not relying on the concavity of W_0 here but: i) $W_0' > 0$ and ii) properties of Bayes' rule.)

We now prove the second statement. Since W_0 is concave and

$$\int \rho'(\pi, \rho) [\rho\sigma_c(\pi) + (1-\rho)\sigma_o(\pi)] d\pi = \rho$$

then

$$W_0(\rho) \geqslant \int W_0(\rho'(\pi, \rho)) [\rho\sigma_c(\pi) + (1-\rho)\sigma_o(\pi)] d\pi$$

Thus randomization can be optimal only if it improves that static outcome by reducing x. Thus, it must be that

$$w(\phi(\pi_{ico}), \pi_{ico}) < \int w(x, \pi) [\rho\sigma_c(\pi) + (1-\rho)\sigma_o(\pi)] d\pi$$

where x is given by (18). A necessary condition is that

$$x < \phi(\pi_{ico}) \iff \mathbb{E}\pi = \int \pi [\rho\sigma_c(\pi) + (1-\rho)\sigma_o(\pi)] d\pi < \pi_{ico}$$

Thus, it is sufficient to show that $\mathbb{E}\pi > \pi_{ico}$ to prove our result. To this end, let

$$\mathbb{E}\pi \geqslant \underline{\pi}(\rho) = \min \int \pi [\rho\sigma_c(\pi) + (1-\rho)\sigma_o(\pi)]$$

subject to $\sigma_c, \sigma_o \in \Delta([0,1])$, (18), (19), and (20). It is then sufficient to show that $\underline{\pi}(\rho) > \pi_{ico}(\rho)$ for ρ close to zero. Suppose by way of contradiction that it is not optimal to have π_{ico} with probability 1 so

$$\int \pi [\rho\sigma_c(\pi) + (1-\rho)\sigma_o(\pi)] d\pi < \pi_{ico} \tag{25}$$

and since we consider $\rho \to 0$ then

$$\mathbb{E}_o \pi = \int \pi \sigma_o(\pi)\, d\pi \leqslant \pi_{ico} \tag{26}$$

otherwise we can make ρ arbitrary close to 0 so that the inequality in (25) is reversed. From the incentive constraint, it must be that $\forall \pi \in \Sigma_o$

$$w(x,\pi) + \beta V_0\left(\rho'(\pi,\rho)\right) \geqslant w(x,\pi^*(x)) + \beta V_0\left(\rho'(\pi^*(x),\rho)\right) \geqslant w(x,\pi^*(x)) + \beta V_0(0) \tag{27}$$

where the second inequality follows from V_0 being increasing in the posterior and $\rho'(\pi^*(x),\rho) \geqslant 0$. Note now that by properties of Bayes' rule

$$\int \rho'(\pi,\rho)\,\sigma_o(\pi)\, d\pi = \int \frac{\rho\sigma_c(\pi)}{\rho\sigma_c(\pi) + (1-\rho)\,\sigma_o(\pi)} \sigma_o(\pi)\, d\pi \leqslant \rho.$$

By strict concavity of w (in π) and V_0 and the fact that ρ is vanishing we have that

$$\begin{aligned}\int \left[w(x,\pi) + \beta V_0\left(\rho'(\pi,\rho)\right)\right]\sigma_o(\pi)\, d\pi &< w(x,\mathbb{E}_o\pi) + \beta V_0\left(\mathbb{E}_o\rho'\right) \\ &\leqslant w(x,\mathbb{E}_o\pi) + \beta V_0(\rho) \\ &= w(\phi(\mathbb{E}_o\pi),\mathbb{E}_o\pi) + \beta V_0(\rho)\end{aligned} \tag{28}$$

Thus combining (27) and (28) we have that

$$w(\phi(\mathbb{E}_o\pi),\mathbb{E}_o\pi) + \beta V_0(\rho) > w(\phi(\mathbb{E}_o\pi),\pi^*(\phi(\mathbb{E}_o\pi))) + \beta V_0(0)$$

Since π_{ico} is the smallest solution to

$$w(\phi(\pi),\pi) + \beta V_0(\rho) \geqslant w(\phi(\pi),\pi^*(\pi)) + \beta V_0(0)$$

then it follows that for ρ close to zero

$$\pi_{ico}(\rho) < \mathbb{E}_o\pi$$

a contradiction. Q.E.D.

A.10 Example where Uncertainty Not Beneficial

Here we present an example of an economy where uncertainty is not beneficial. This economy is similar to our bailout example but there is *no effort choice*.

There are two types of private agents: depositors and bankers. At the beginning of each period the banker has no capital and must borrow $k = 1$ from the depositors to finance an investment opportunity that pays off at the end of the period. The return of the investment opportunity is R_H with probability p and 0 with probability $1-p$. The banker offers a contract to depositors that promises to repay R units of the consumption good in the second sub-period subject to limited liability. We assume that there are bankruptcy costs ψ in case of a default. The policy maker can avoid this bankruptcy costs by making a transfer to the banker. In particular, the policy maker can choose the recovery rate π in case of inability of the banker to repay. With a recovery rate π, the bankruptcy costs are $\psi(1-\pi)$.

The rule designer and the policy makers care about the lenders' utility net of bankruptcy costs.

Depositors are then willing to lend to the banker at an interest rate

$$Q(b, \mathbb{E}\pi) = [p + (1-p)\mathbb{E}\pi]$$

so the banker's problem is

$$\max_b p \max\{R_H - b, 0\}$$

subject to

$$1 = Q(b, \mathbb{E}\pi)\, b.$$

In equilibrium, it must be that $b = 1/Q(b, \mathbb{E}\pi)$ so we can let $x = Q$ with

$$Q = \phi(\mathbb{E}\pi) = [p + (1-p)\mathbb{E}\pi] \tag{29}$$

and express the social welfare function as

$$w(Q, \pi) = -1 + p\frac{1}{Q} - (1-p)\psi\frac{1}{Q}\max\{1-\pi, 0\}$$

We first show that the solution to the static rule designer's value is convex in ρ:

Lemma 7. *If $p > 1/2$, the static rule designer's value $W_0(\rho)$ is convex.*

Proof. Consider

$$W_0(\rho) = \max_{\pi_c, Q} -\left[1 - p\frac{1}{Q}\right] - (1-p)\rho\psi\frac{1}{Q}\max\{1-\pi_c, 0\}$$

subject to (29) or, using the constraint to substitute for Q as

$$W_0(\rho) = \max_{\pi_c} -1 + \frac{[p - (1-p)\rho\psi(1-\pi_c)]}{[p + (1-p)(\rho\pi_c + (1-\rho))]}$$

Differentiating with respect to π_c we obtain

$$\frac{(1-p)\,\rho\psi\,[p+(1-p)\,(\rho\pi_c+(1-\rho))]-(1-p)\,\rho\,[p-(1-p)\,\rho\psi\,(1-\pi_c)]}{[p+(1-p)\,(\rho\pi_c+(1-\rho))]^2}$$

which is negative if ψ is sufficiently small. So the optimal static rule is $\pi_c = 0$ for all ρ. Thus we have that

$$\begin{aligned} W_0(\rho) &= -1+\frac{[p-(1-p)\,\rho\psi]}{[p+(1-p)\,(1-\rho)]} \\ &= \frac{[p-(1-p)\,\rho\psi]}{p-\rho\,(1-p)}-1 \end{aligned}$$

which is convex as

$$\begin{aligned} W_0'(\rho) &= \frac{-(1-p)\,\psi\,[p-\rho\,(1-p)]+(1-p)\,[p-(1-p)\,\rho\psi]}{[p-\rho\,(1-p)]^2} \\ &= (1-p)\,\frac{[p-(1-p)\,\rho\psi]-\psi\,[p-\rho\,(1-p)]}{[p-\rho\,(1-p)]^2} \\ &= (1-p)\,\frac{p-\psi p}{[p-\rho\,(1-p)]^2} \\ &= \frac{(1-p)\,p\,(1-\psi)}{[p-\rho\,(1-p)]^2} \end{aligned}$$

and

$$W_0''(\rho) = 2\,(1-p)\,\frac{(1-p)\,p\,(1-\psi)}{[p-\rho\,(1-p)]^3} = 2\,(1-p)\,\frac{(1-p)\,p\,(1-\psi)}{[p\,(1+\rho)-\rho]^3} > 0$$

as long as $p > 1/2$. □

Consider now the dynamic problem (twice repeated) in (8). Because of the convexity of $W_0(\rho)$, the dynamic benefits of pooling, $\Delta\Omega(\rho) = W_0(\rho) - [\rho W_0(1) + (1-\rho)\,W_0(0)]$, are negative. The static benefits of pooling, $\Delta\omega$, are positive for low levels of reputation and negative for high levels. If β is sufficiently high we have that $\Delta\omega(\rho) + \beta\Delta\Omega(\rho) < 0$ and so we have the following counterpart of Proposition 1:

Proposition 8. *In this example where* condition (5) *does not hold and uncertainty is welfare reducing, under Assumptions 1 and 3, if* β *is sufficiently large then for all* ρ *there is separation with probability 1 and* $\pi = \pi_0$.