Efficient Sovereign Default *

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Abstract

In this paper, I show that the key aspects of sovereign debt crises can be rationalized as part of the efficient risk-sharing arrangement between a sovereign borrower and foreign lenders in a production economy with informational and commitment frictions. The constrained efficient allocation involves ex post inefficient outcomes that resemble sovereign default episodes in the data and can be implemented with non-contingent defaultable bonds and active maturity management. Defaults and periods of temporary exclusion from international credit markets happen along the equilibrium path and are essential to supporting the efficient allocation. Furthermore, during debt crises, the maturity composition of debt shifts toward short-term debt and the term premium inverts as in the data.

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1 Introduction

The conventional framework to study sovereign debt crises is the incomplete-market approach that follows the seminal contribution of Eaton and Gersovitz (1981). Through the lens of these models, inefficiencies are pervasive since defaults are due to incomplete contracts. Moreover, the high reliance on short-term debt is often blamed as one of the main causes of sovereign or external debt crises.[1] Models with roll-over risk (e.g., Cole and Kehoe (2000)) provide a rationale for such a prediction, because short-term debt is more prone to rollover risk.

This paper takes a first step toward bridging the gap between the literature on quantitative incomplete markets and the literature on constrained efficient risk-sharing arrangement. I show that the key aspects of sovereign debt crises can be rationalized as part of the efficient risk-sharing arrangement between a sovereign borrower and foreign lenders in a production economy with informational and commitment frictions. The efficient allocation can be implemented as the equilibrium outcome of a *sovereign debt game* with non-contingent defaultable debt of multiple maturities. Defaults and ex post inefficient outcomes along the equilibrium path are not a pathology; rather, they support the ex ante efficient outcome. Moreover, the equilibrium outcome path displays defaults when output is low, an inversion of the term structure of interest rates spreads when the spread level is high, and a negative association between the duration of outstanding debt and the probability of future defaults as documented in the data.

I provide a different view on policies than the literature on incomplete markets. First, the negative correlation between the maturity of outstanding debt and the probability of a crisis emerges as a way to support the efficient allocation when only non-contingent defaultable debt of multiple maturities is available. Hence, the high reliance on short-debt debt is just a symptom, and not a cause, of an imminent debt crisis in contrast to the view in Cole and Kehoe (2000). Second, dilution of long-term debt in the absence of seniority clauses is essential to replicate the insurance prescribed by the efficient allocation given the available assets. Hence, introducing and enforcing seniority clauses will lower welfare. This contrasts with the predictions in Chatterjee and Eyigungor (2012) and Hatchondo et al. (2012). Finally, because ex post inefficient outcomes are part of the efficient allocation, interventions by a supranational authority aimed at reducing the inefficiencies in a sovereign default episode are not beneficial from an ex ante perspective.

I consider a simple production economy in which imported intermediate inputs are used in production, similarly to Aguiar et al. (2009) and Mendoza and Yue (2012). The government cannot commit to repaying its debt, and the only recourse available to the

[1] For instance, Rodrik and Velasco (1999) find that the ratio of short-term debt to reserves is a robust predictor of an external debt crisis.

foreign lenders to ensure repayment is the threat of exclusion from future borrowing and trade. Moreover, the government has some private information about the state of the domestic economy.[2] In the baseline economy, the source of private information is the relative productivity of the domestic non-tradable sector.[3] One interpretation of this assumption is that the government has more information about the domestic economy than do foreign lenders, and it controls the released statistics.[4]

I first characterize the optimal risk-sharing arrangement between the government and the foreign lenders subject to the restrictions imposed by the lack of commitment and private information. Absent contracting frictions, the risk-neutral lenders would completely insure the risk-averse government against fluctuations in productivities, and the realization of the shock would have no effect on the continuation of the allocation. Both the private information and the government's lack of commitment limit such insurance. In particular, because of the presence of private information, the provision of a dynamic incentive is needed to have insurance. In order for insurance payments to borrowers with currently low-productivity-shocks to be incentive compatible, there must be a cost associated with claiming to have low productivity. Lenders can impose such a cost by reducing the continuation value of the government through lowering its future consumption levels.

The lack of commitment interacts with the incentive problem. In particular, when the government's continuation value is low, the government is tempted to deviate from the efficient allocation by increasing current consumption by not repaying lenders and then living in autarky thereafter. To prevent such an outcome, the lenders must provide a sufficiently low amount of intermediate goods so that this kind of deviation is unprofitable. Enforcing a continuation value for the government close to autarky is ex post inefficient. That is, if the government and the lenders could renegotiate the terms of their agreement, committing not to do so again in the future, then both could be made better off. By increasing the government's value when it is close to autarky, it is possible to avoid the drop in imported intermediate inputs, which depresses production and reduces the government's ability to repay the lenders. The necessity of providing incentives ex ante requires that these ex post inefficient outcomes happen along the equilibrium path with strictly positive probability.

I show that under appropriate sufficient conditions, any efficient allocation settles down to a stationary distribution with ex post inefficiencies. The theme that ex post

[2]The main results of the paper remain valid if I consider an environment in which the incentive problem arises because of moral hazard, as in Atkeson (1991) and Tsyrennikov (2013).

[3]In the Appendix, I show that this formulation is isomorphic to one in which the source of private information is a *taste shock* that affects the marginal utility of consumption of the borrower, as in Atkeson and Lucas (1992).

[4]See Rogoff (2011) for such an argument.

inefficiencies along the equilibrium path are necessary to support the ex ante optimal arrangement in economies with incentive problems has been explored in various contexts (e.g.,Green and Porter (1984), Phelan and Townsend (1991), and Yared (2010)). Quadrini (2004), Clementi and Hopenhayn (2006), DeMarzo and Sannikov (2006), DeMarzo and Fishman (2007), and Hopenhayn and Werning (2008) study this in the context of firm dynamics with credit frictions. A novel feature of my paper is that there is no termination of the risk-sharing relationship. The optimal outcome has periods of temporary autarky (which are ex post inefficient), but cooperation eventually restarts after the domestic economy recovers and the government makes a payment to the lenders.

I then turn to implementing an efficient allocation as an equilibrium outcome of a sovereign debt game, similar to what is considered by the literature on quantitative incomplete markets. The set of assets that the government can issue is restricted to non-contingent bonds of multiple maturities. The government has the option to *default*, which I define as suspending the principal and coupon payments specified by the bond contracts. The government is excluded from credit markets until a given partial repayment to the bondholders is made. The government can also impose a tax on the payments received by foreign exporters from the domestic firms for the imports of intermediate goods, capturing the idea that the government cannot commit to repay the foreign lenders.

Along the equilibrium outcome path, there are defaults only when the government's continuation value is equal to the autarkic value. In this sense, defaults in the model are infrequent. Moreover, defaults are associated with high indebtedness (relative to the maximal level of debt sustainable), low output, and ex-post inefficiencies.

The crucial step in proving that the efficient allocation can be an outcome of the sovereign debt game is to show that it is possible to replicate the wealth transfers implied by the efficient allocation with non-contingent defaultable debt. Defaults and partial repayments introduce de facto implicit state contingencies in the bond contract. When there is full repayment, the state contingent returns implied by the efficient allocation are replicated by exploiting the variation in the price of long-term debt after a shock. After the realization of a low productivity shock, the continuation value for the sovereign borrower decreases and the probability that there will be a default in the near future goes up. The increase in the likelihood of a future default reduces the value of the outstanding long-term debt. This reduction results in a capital loss for the lenders and provides some debt relief to the borrower after an adverse shock. The opposite happens after the realization of a high productivity shock: the price of outstanding long-term debt goes up and the lenders realize a capital gain. If the maturity composition of debt is appropriately chosen, this mechanism can replicate the state-contingent returns implied by the efficient allocation.

The model can generate the features of output, consumption, imports, and exports that occur during and after debt crises. The proximate cause of a debt crisis is a sufficiently long string of low productivity shocks, which lead the borrower's continuation value to decrease until it reaches the value of autarky, when there is a default. The lack of commitment implies that the imports of intermediates must drop to prevent a deviation by the borrower. This drop in imports reduces output, consumption, and the payments made to the lenders. Once the economy receives a high productivity shock in the non-tradable sector, output increases and the borrower runs a trade surplus to partially repay the defaulted debt. These repayments result in the gradual increase of the borrower's continuation value; hence, consumption, production, and imported intermediate inputs used in production will also increase.

Along the path approaching default, the maturity composition of the sovereign debt shifts toward short-term debt. This shift occurs because, when the probability of future default is high, the price of the long-term debt is more sensitive to shocks. Therefore, a lower long-term debt holding is needed to replicate the debt relief that is implicit in the efficient allocation after a low realization of productivity in the non-tradable sector. Since the overall indebtedness of the sovereign borrower is increasing along the path approaching a default, it must be that the amount of short-term debt issued goes up as the probability of default increases. Thus, the maturity composition shortens as indebtedness increases. Furthermore, during debt crises, the term spread curve inverts, as documented by Broner et al. (2013) and Arellano and Ramanarayanan (2012) for emerging economies.

Related Literature This paper is related to several strands of literature. First, it is related to the literature on quantitative incomplete markets on sovereign default. Following Eaton and Gersovitz (1981), recent contributions include Aguiar and Gopinath (2006), Arellano (2008), Hatchondo and Martinez (2009), Benjamin and Wright (2009), Yue (2010), Arellano and Ramanarayanan (2012), Chatterjee and Eyigungor (2012), and Mendoza and Yue (2012). I depart from this literature by studying the best equilibrium outcome of the sovereign debt game when the maturity structure is sufficiently rich. As previously argued, the policy implications that arise from assuming that markets are incomplete differ substantially from those that arise from allowing for the possibility of complete markets.

My work is also related to the literature on optimal contracting approach to sovereign borrowing (e.g.,Atkeson (1991), Thomas and Worrall (1994), Kehoe and Perri (2002), Kehoe and Perri (2004), Aguiar et al. (2009), and Tsyrennikov (2013)). More broadly, this paper is also related to the literature on dynamic contracting with informational and commitment frictions (e.g., Atkeson and Lucas (1995) and Ales et al. (2014)). Green (1987), Thomas and Worrall (1990), and Atkeson and Lucas (1992) consider environments with only private information, while Kehoe and Levine (1993), Kocherlakota (1996), and Albuquerque and Hopenhayn (2004) consider only lack of commitment. The main contribution of my paper relative to this strand of the literature is to propose an implementation that relates the efficient outcome to the data series that are the focus of the empirical research: interest rates, default decisions, and maturity composition of debt. In this sense, this paper takes a first step toward bridging the gap between the literature on quantitative incomplete markets and that on constrained efficient arrangements.[5]

The idea that managing the maturity composition of debt serves to replicate state contingent returns has been explored, among others, by Kreps (1982), Angeletos (2002), and Buera and Nicolini (2004). In these models, movements in the term structure of interest rates are generated by fluctuations in the equilibrium stochastic discount factor. I consider a small open economy environment with risk-neutral lenders where the international interest rate is orthogonal to the shocks in the domestic economy. The variation in price of long-term debt is generated through variation in the endogenous probability of future default. In this aspect, my paper is closely related to the work of Arellano and Ramanarayanan (2012), who endogenize the time-varying maturity composition of debt in an Eaton and Gersovitz (1981) type of model. Consistent with my findings, these authors find that the maturity composition of debt shortens when the probability of default is high. The difference between their paper and mine is that they cannot assess the efficiency of such an equilibrium outcome. Moreover, they determine the maturity composition by trading off a hedging motive and a commitment-not-to-dilute motive. The hedging motive is that long-term debt is attractive because it allows for a state contingent return. The commitment-not-to-dilute motive is that short-term debt is a commitment device not to dilute outstanding long-term debt. In my paper, the hedging motive alone is sufficient to shorten the duration of debt when the probability of default is high.

The rest of the paper is organized as follows. In Section 2, I present a set of stylized facts about sovereign borrowing and lending. while i In Section 3, I describe the environment, while in Section 4, I characterize the efficient allocation. In Section 5, I define the sovereign debt game. In Section 6, I construct and characterize the default rule, bond prices, and holdings that support the efficient allocation as an equilibrium outcome of the sovereign debt game. Finally, after discussing the implementation in Section 7, I conclude the paper in Section 8.

[5]The implementation proposed in this paper works with several models cited above and can be applied not only to sovereign debt but also to firm-level dynamics. It is an alternative to the one developed by DeMarzo and Fishman (2007) and DeMarzo and Sannikov (2006).

2 Stylized Facts on Sovereign Borrowing

This section summarizes an extensive literature that documents the behavior of output, consumption, and imports of intermediate goods around sovereign default episodes and other key aspects of the data about sovereign borrowing and lending. For an excellent survey of the empirical literature, see Tomz and Wright (2013). The simple model I construct aims to qualitatively account for such facts.

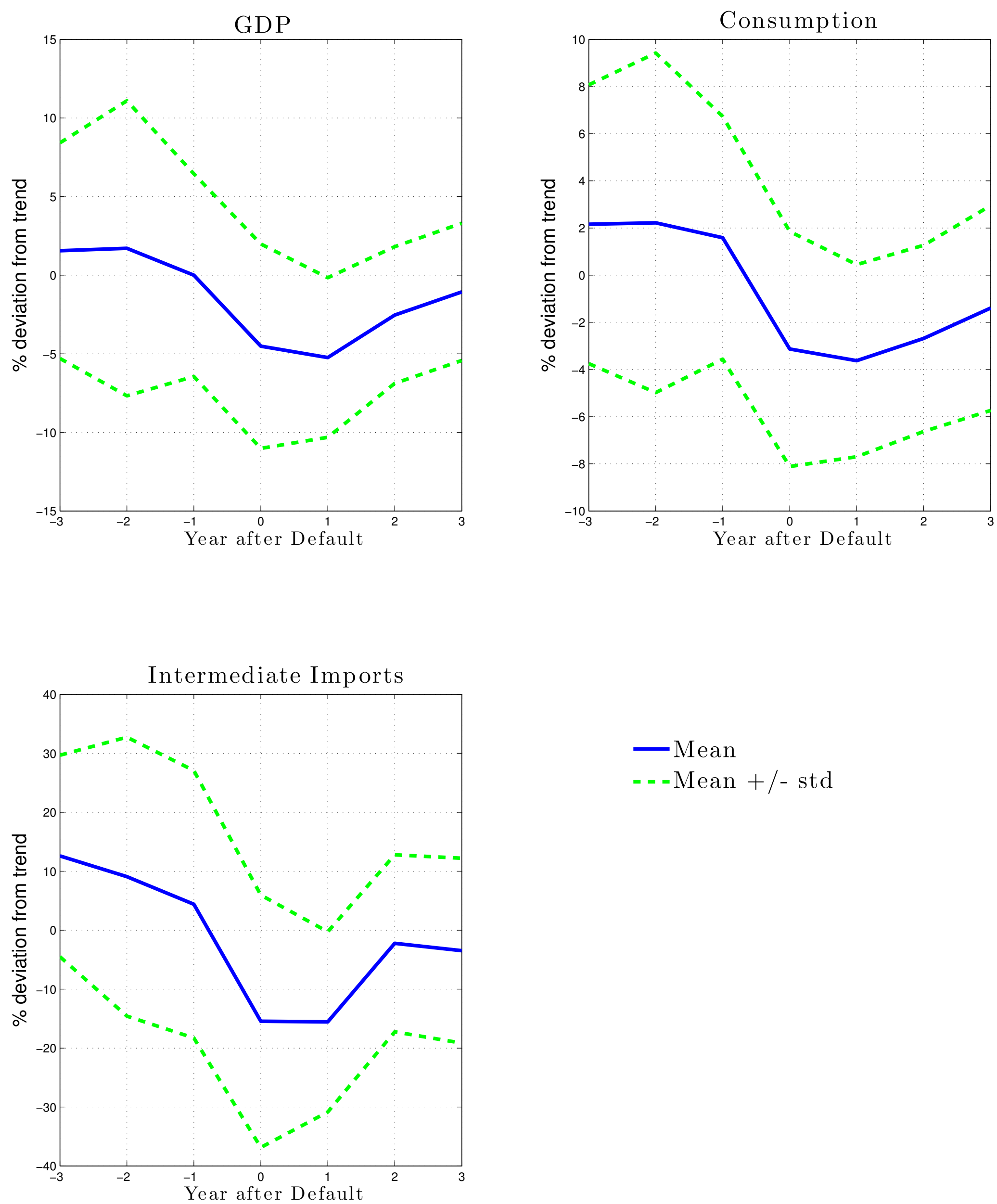
Aggregates around sovereign default episodes. Sovereign defaults are infrequent events. In a dataset spanning from 1829 to present, Tomz and Wright (2007) find that the unconditional probability of a borrower defaulting on its debt is less than 2%. Figure 1 shows the average behavior of economic aggregates around episodes of sovereign default in the 1980s, 1990s, and 2000s. Details about the data and the sample of 23 default episodes are provided in the Appendix. The first two panels of Figure 1 illustrate the dynamics for GDP and consumption around a sovereign default episode. On average, in the 23 default episodes considered, output is 4.5% and 5.2% below trend and consumption is 3.1% and 3.6% below trend in the year of the default and the year after, respectively. This pattern confirms what has been documented in several studies (see for example, Mendoza and Yue (2012) and the references in the survey by Panizza et al. (2009)).

Tomz and Wright (2007) is a notable exception. They find only a weak association between default episodes and output being below trend. In a larger sample, they show that about a third of default episodes happen with output above trend. This observation constitutes a problem for models where the only shock is output or productivity, as the one in this paper. Allowing for additional sources of shocks can help in this regard.[6]

A large literature documents that sovereign default episodes are accompanied by large drops in trade. For instance, see Rose (2005) and the references in the survey by Panizza et al. (2009). Also, Borensztein and Panizza (2009) document that a default has a negative impact on trade credit. Arteta and Hale (2008) show that foreign credit to the private sector collapses in the aftermath of a default, and Fuentes and Saravia (2010) show that defaults lead to a fall in foreign direct investment flows into the country. As noted in Mendoza and Yue (2012), the drop in imports of intermediate goods is very large: imports drop on average from 4.4% above trend the year before a default to about 15.5% below trend the year of a default and the following year. See the third panel in Figure 1. This drop is larger than in recessions of similar magnitudes. A simple regression shows that the drop in intermediate inputs in the year of and the year following a sovereign

[6]The model presented below is isomorphic to an economy with taste shock to the marginal utility of consumption. If one adds productivity shocks in such an environment, one can generate defaults in periods when output is above trend when the exogenous productivity is high but the high urgency to consume can trigger a default.

Figure 1: Dynamics of aggregates around default episodes.



Details about data are provided in the Appendix.

default is more than 10% larger than what one would expect from a drop in output of the same magnitude, absent default. See the Appendix for further details. This drop can have a non-trivial impact on the economy. Gopinath and Neiman (2014) present a model calibrated to replicate the crisis in Argentina in 2002 and show that the decline in imports of intermediate goods can account for up to a 5 percentage point decline in the welfare-relevant measure of productivity. More generally, Hébert and Schreger (2015) use a clever identification to argue that external defaults generate large costs in terms of output losses.

Behavior of maturity composition of debt and term structure. The dynamics of the maturity composition of debt and the term structure of interest rate spreads play a critical role in the implementation I propose. It is then important to contrast the model's implications with the data on maturity choices and term structure. Broner et al. (2013) use data from 11 emerging economies from 1990 to 2009 to document that during emerging market debt crises (when spreads are high), the maturity of newly issued debt shortens, as confirmed by Arellano and Ramanarayanan (2012). These findings are about new issuance, while the results in the model are about the stock of outstanding debt. If new issuances are shorter than average, then also the average duration of the stock of debt will tend to decrease. Recent studies provide direct evidence for the model's prediction. For the case of Argentina, Buera and Nicolini (2010) document that the stock of outstanding debt was more tilted toward shorter maturity in 2000 (at the onset of the default in January of 2002) relative to 1997 (see Figure 5 in their paper). Bocola and Dovis (2016) provide similar evidence for Italy in the current crisis.

Finally, consider the typical behavior of the term structure of interest rate spreads. Broner et al. (2013) and Arellano and Ramanarayanan (2012) document that when the spreads are low, long-term spreads are generally higher than short-term spreads. During debt crises, the gap between long- and short-term spreads tends to narrow and the term spread curve flattens or even inverts.

Next, I will construct a simple model and show that the efficient allocation can be decentralized in a way that is consistent with these findings. In particular, sovereign debt crises are associated with trade disruption and a drop in output, and during debt crises, defined as periods of high interest rate spreads, the maturity composition of debt is more tilted toward short-term debt and the yield curve inverts.

3 Environment

In this section, I lay out the environment in which the source of private information is the productivity of the non-tradable sector. In the Appendix, I provide a reinterpretation of this economy as a taste shock economy, as in Atkeson and Lucas (1992).[7]

Time is discrete and indexed by $t = 0, 1, ...$ There are three types of agents in the economy: a large number of homogeneous domestic households, a benevolent domestic government, and a large number of foreign lenders. There are three types of goods: a domestic consumption good (non-traded), an export good, and an intermediate good. The source of uncertainty is a shock to the relative productivity of the domestic consumption (non-tradable) sector.

Preferences All agents are infinitely lived. The stand-in domestic household values a stochastic sequence of consumption of the domestic good, $\{c_t\}_{t=0}^{\infty}$, according to

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(c_t), \tag{1}$$

where $\beta \in (0, 1)$ is the discount factor, and the period utility function has constant relative risk aversion,

$$U(c) = \frac{c^{1-\gamma}}{1-\gamma}, \tag{2}$$

with $\gamma > 1$. The government is benevolent, and it maximizes the utility of the stand-in domestic household.

Foreign lenders are risk neutral, and they value consumption of the export good. They discount the future with a discount factor $q \in (0, 1)$, which should be thought of as the inverse of the risk-free interest rate in international credit markets. I allow the discount factor β and q to differ, but I will restrict myself to the case where $q \geqslant \beta$; that is, the domestic households discount the future at a weakly higher rate than the international interest rate.

Endowments and Technology Foreign lenders have a large endowment of the intermediate good. They have access to a technology that transforms one unit of the intermediate good into one unit of the export good so the relative price between the export and the intermediate good is fixed at one.

Each domestic household is endowed with one unit of labor in each period. There is a domestic production technology that transforms the intermediate good and labor into

[7]Under this interpretation, the link between output and default is weakened and so it may help to account for the weak relationship found in the data by Tomz and Wright (2007).

the domestic consumption good, c, and foreign consumption good, x, as follows:

$$c \leqslant zF(m_1, \ell_1) \quad \text{and} \quad x \leqslant F(m_2, \ell_2) \tag{3}$$

$$m_1 + m_2 \leqslant m, \quad \ell_1 + \ell_2 \leqslant 1, \tag{4}$$

where m_1 and m_2 are the units of the intermediate good allocated to the production of the domestic and export good, respectively; m is the total amount of intermediates used domestically; and ℓ_1 and ℓ_2 are the units of domestic labor allocated to domestic and export production, respectively. The production function, $F : \mathbb{R}$, has constant returns to scale; it is increasing and continuously differentiable, it satisfies the Inada condition $\lim_{m \to 0} F_m(m, \ell) = +\infty \ \forall \ell > 0$, and it is such that $F(0, 1) > 0$, so strictly positive output can be produced in autarky. For notational convenience, let $f(m) = F(m, 1)$. The relative productivity of the domestic sector, z, is distributed according to a probability distribution μ, and it is i.i.d. over time. For simplicity, let z take on only two values, $z \in \{z_L, z_H\}$ with $z_L < z_H$. Due to the properties of constant-returns-to-scale technology, the technological restrictions imposed by (3)-(4) can be summarized by the following aggregate resource constraint

$$\frac{c}{z} + x \leqslant f(m), \tag{RC}$$

as well as the non-negativity conditions on c and x.

Timing The timing of events within the period is as follows:

1. Foreign lenders supply intermediate goods $m_t \geqslant 0$;
2. the productivity shock z_t is realized according to μ; and
3. real activity occurs: production, consumption, and exporting take place.

Let $z^t = (z_0, z_1, ..., z_t)$. An allocation is a stochastic process $\mathbf{x} \equiv \{m(z^{t-1}), c(z^t), x(z^t)\}_{t=0}^{\infty}$. An allocation $\mathbf{x}$ is feasible if it satisfies the resource constraint (RC) for all t, z^t.

Information Foreign lenders observe the amount of intermediate goods that the country imports, m, and the amount of exports, x. Moreover, they can observe the amount of resources, m_1 and ℓ_1, employed in the domestic consumption (non-tradable) sector. They cannot see the amount of output produced with the inputs, because the realization of z is privately observed by the domestic government. From (RC), foreign lenders can use their information about m and x to infer c/z but not c and z separately.

I collect assumptions made so far here:

Assumption 1 *The utility function is* $U(c) = c^{1-\gamma}/(1-\gamma)$ *with* $\gamma > 1$*; the discount factor* β *satisfies* $qz_H^{1-\gamma}/\mathbb{E}\left(z^{1-\gamma}\right) \leqslant \beta \leqslant q$*; and the production function* $F(m, \ell)$ *is increasing, continuously differentiable, displays constant returns to scale,* $F(0, 1) > 0$*, and satisfies the condition* $\lim_{m \to 0} F_m(m, \ell) = +\infty \ \forall \ell > 0$.

I assume that $\gamma > 1$ to ensure that the government has a large incentive to walk away from the risk sharing arrangement after low productivity shock. The opposite happens when $\gamma < 1$. The condition $\beta > qz_H^{1-\gamma}/\mathbb{E}\left(z^{1-\gamma}\right)$ ensures that the government is sufficiently patient so that in the high productivity state it prefers to reduce its consumption below the autarky level, being rewarded with an increase in future consumption.

4 Efficient Allocation

In this section, I define and characterize a (constrained) efficient allocation when lenders cannot separately observe c and z, and the sovereign borrower cannot commit to repay. I establish that, under certain sufficient conditions, an efficient allocation has cyclical periods with ex post inefficient outcomes that resemble a sovereign default episode in the data.

4.1 Definition

Private information and lack of commitment by the sovereign borrower impose constraints in addition to the resource constraint (RC), which an allocation must satisfy in order to be implementable.

First, consider the restriction imposed by the fact that z is privately observed by the borrower. By the *revelation principle*, it is without loss of generality to focus on the direct revelation mechanism in which the sovereign borrower reports his type. Define the continuation utility for the sovereign borrower associated with the allocation $\mathbf{x}$ after history z^t (according to truth-telling) as

$$(5) \qquad v(z^t) \equiv \sum_{j=1}^{\infty} \sum_{z^{t+j}} \beta^{j-1} \Pr(z^{t+j}|z^t) U(c(z^{t+j})).$$

An allocation $\mathbf{x}$ is incentive compatible if and only if it satisfies the following (temporary) *incentive compatibility constraint* for all t, z^t, z:

$$(\text{IC}) \qquad U(c(z^{t-1}, z_t)) + \beta v(z^{t-1}, z_t) \geqslant U\left(z_t\left[f(m(z^{t-1})) - x(z^{t-1}, z)\right]\right) + \beta v(z^{t-1}, z)$$

where $z_t\left[f(m(z^{t-1})) - x(z^{t-1}, z)\right]$ is the consumption of non-traded good if the borrower has productivity z_t but exports the amount $x\left(z^{t-1}, z\right)$ instead of $x\left(z^{t-1}, z_t\right)$. The incentive compatibility constraint, (IC), captures the informational frictions in the economy. It ensures that the borrower has no incentive to engage in *undetectable deviations*. That is, after any history z^t, the borrower does not want to choose the action prescribed for type $z \neq z_t$.

Second, consider the restrictions imposed by lack of commitment. To be implementable, an allocation $\mathbf{x}$ must satisfy the following *sustainability constraint* for all t, z^t:

$$\text{(SUST)} \qquad U(c(z^t)) + \beta v(z^t) \geqslant U(z_t f(m(z^{t-1}))) + \beta v_a,$$

where v_a is the value of autarky given by

$$\text{(6)} \qquad v_a \equiv \frac{\sum_z \mu(z) U(z f(0))}{1-\beta}.$$

The sustainability constraint, (SUST), requires that the borrower have no strict incentive to engage in *detectable deviations*. That is, after any history, the borrower cannot gain from increasing his consumption by failing to export $x(z^t)$. As it is standard in the literature, I assume that after this detectable deviation, the borrower is punished with autarky. This entails two forms of punishment. First, the sovereign borrower cannot access credit markets to obtain insurance. Second, the borrower suffers a loss in production because he cannot use imported intermediate goods. Later, I will show this autarkic value is the worst equilibrium value of the sovereign debt game.

A feasible allocation $\mathbf{x}$ is said to be *efficient* if it maximizes the present value of net transfers to the foreign lenders, $x - m$, subject to the resource constraint (RC), the incentive compatibility constraint (IC), the sustainability constraint (SUST), and a participation constraint for the borrower,

$$\text{(PC)} \qquad \sum_{t=0}^{\infty} \sum_{z^t} \beta^t \Pr(s^t) U(c(z^t)) \geqslant v_0$$

for some feasible initial level of promised utility $v_0 \in [v_a, \bar{v}]$, with $\bar{v} \equiv \lim_{c\to\infty} U(c)/(1-\beta)$. An efficient allocation solves

$$\text{(J)} \qquad J(v_0) = \max_{\mathbf{x}} \sum_{t=0}^{\infty} \sum_{z^t} q^t \Pr(s^t) \left[x(z^t) - m(z^{t-1})\right]$$

subject to (RC), (IC), (SUST), and (PC). I will refer to the value $J : [v_a, \bar{v}] \to \mathbb{R}$ as the *Pareto frontier*.

The constraint set in (J) is not necessarily convex, because of the presence of $U \circ f(m)$, a concave function, on the right-hand side of the sustainability constraint (SUST). Thus, randomization may be optimal. It is possible to rule out randomization as part of the efficient allocation by making an additional assumption following Aguiar et al. (2009).

Assumption 2 *Define* $H : [U(f(0)), U(f(m^*))] \to \mathbb{R}$ *as* $H(\underline{u}) \equiv C(\underline{u}) - f^{-1} \circ C(\underline{u})$ *with* $C = U^{-1}$. H *is concave.*

If Assumption 2 is satisfied,[8] then randomization is not optimal.

4.2 Near-Recursive Formulation

The problem in (J) admits a near-recursive formulation using the borrower's promised utility, v, as a state variable. From $t \geqslant 1$, an efficient allocation solves the following recursive problem for $v \in [v_a, \bar{v}]$:

$$B(v) = \max_{m, c(z), v'(z)} \sum_z \mu(z) \left[f(m) - m - c(z) + qB(v'(z)) \right] \tag{P}$$

subject to

$$\begin{aligned}
U(c(z)) + \beta v'(z) &\geqslant U\left(z\left[f(m) - y^*(z')\right]\right) + \beta v'(z') \quad \forall z, z' &(7)\\
U(c(z)) + \beta v'(z) &\geqslant U(zf(m)) + \beta v_a \quad \forall z &(8)\\
v'(z) &\geqslant v_a \quad \forall z &(9)\\
\sum_z \mu(z) \left[U(c(z)) + \beta v'(z) \right] &= v, &(10)
\end{aligned}$$

where $B(v)$ is the maximal present discounted value of net transfers, $x - m = f(m) - c - m$, that the foreign lenders can attain subject to a recursive version of the incentive compatibility constraint, (7), a recursive version of the sustainability constraint, (8), the fact that continuation utility must be greater than the value of autarky, (9), and the constraint that the recursive allocation delivers a value of v to the sovereign borrower (the *promise-keeping constraint*), (10). The function B traces out the *utility possibility frontier*.

At $t = 0$, for all $v_0 \in [v_a, \bar{v}]$, the problem in (J) can be expressed as

$$J(v_0) = \max_{m, c(z), v'(z)} \sum_z \mu(z) \left[f(m) - m - c(z) + qB(v'(z)) \right] \tag{11}$$

subject to (7), (8), (9), and the participation constraint

$$\sum_z \mu(z) \left[U(c(z)) + \beta v'(z) \right] \geqslant v_0. \tag{12}$$

[8] Assumption 2 is satisfied if the curvature in U and f is low.

The difference between the Pareto frontier J and the utility possibility frontier B is that, in the utility possibility frontier, the promise-keeping constraint (10) requires that the allocation deliver *exactly* the promised utility $v \in [v_a, \bar{v}]$ to the sovereign borrower. This is because for $t \geqslant 1$, the promise-keeping constraint serves to maintain incentives from previous periods. In contrast, in the definition of the Pareto frontier J, the participation constraint (12) requires that the sovereign borrower receive *at least* v. This is because in period $t = 0$ there are no incentives from previous periods to keep. In many applications, this asymmetry is irrelevant, because the participation constraint in (J) is binding. This is not the case here, because the utility possibility frontier $B(v)$ has an increasing portion, as I will later show.

4.3 Properties

The next proposition establishes three properties of the efficient allocation that I will later use to characterize the equilibrium outcome that supports the efficient allocation.

Proposition 1 *Under Assumptions 1 and 2, the efficient allocation is such that*

- *i) There are distortions in production. Let m^* be the statically efficient level of intermediates, i.e., m^* such that $f'(m^*) = 1$. There exists $v^* \in (v_a, \bar{v})$ such that $m(v) = m^*$ for all $v \geqslant v^*$. For all $v \in [v_a, v^*)$, $m(v)$ is strictly less than m^* and is strictly increasing in v, and, in particular, $m(v_a) = 0$.*

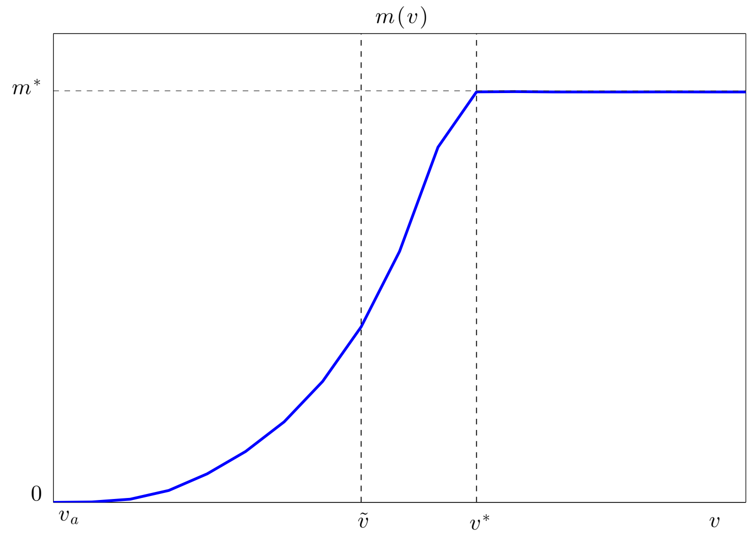
- *ii) The efficient allocation is dynamic: $\forall v \in [v_a, \bar{v}]$, $c(v, z_L) > c(v, z_H)$ and $v'(v, z_L) < v'(v, z_H)$.*

- *iii) There is insurance. Let $b(v, z) \equiv x(v, z) - m(v) + qB(v'(v, z))$ be the lenders' value after the realization of z. Then $\forall v \in [v_a, \bar{v}]$, $b(v, z_H) > b(v, z_L)$.*

- *iv) The value for the lenders when $v = v_a$ is positive, $b(v_a, z) > 0$ for all z.*

The proof of this proposition can be found in the Appendix. Part i) states that low levels of promised utility for the borrower are associated with imported intermediates that are below the statically efficient level, m^*, which is such that $f'(m^*) = 1$. When the continuation value for the borrower is low, imports must be low to satisfy the sustainability constraint. Whenever the sustainability constraint is binding, $m < m^*$. In particular, at autarky it must be that $m(v_a) = 0$. In fact, if the foreign lenders supplied any $m > 0$, the sovereign government could unilaterally achieve a lifetime utility of $U(zf(m)) + \beta v_a > U(zf(0)) + \beta v_a = v_a$. Thus, only $m = 0$ is consistent with the promise-keeping and sustainability constraints at autarky. On the other hand, for continuation values high enough, $v \geqslant v^*$, the threat of autarky after an observable deviation

Figure 2: Policy function for intermediate imports.



is sufficiently harsh that the statically efficient amount of intermediate imports can be supported; that is, $m(v) = m^*$ for all $v \geqslant v^*$. It can be shown that m is actually strictly increasing in the borrower's promised value for $v \in [v_a, v^*]$. This result is illustrated in the first panel of Figure 2.

Part ii) states that the efficient allocation is *dynamic*, in the sense that it uses variation in the borrower's continuation utility to provide incentives, thus allowing for higher transfers after the realization of a low productivity shock. This feature of the efficient allocation is critical for it to be implementable as an outcome of the sovereign debt game.

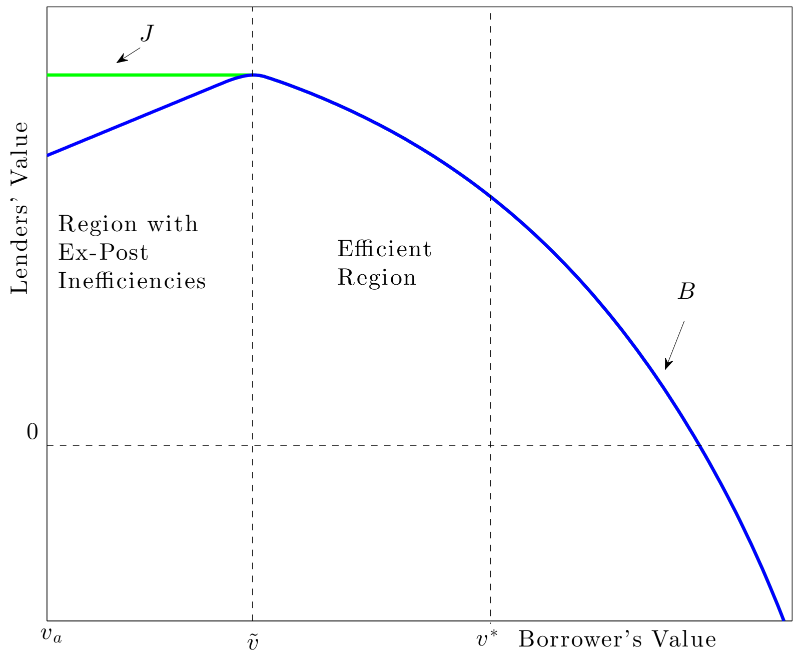
Part iii) shows that the market value of debt is state-contingent; there is debt relief when the borrower has a high marginal utility of consumption (low z). Thus, the efficient allocation provides some, albeit imperfect, insurance.

Part iv) states that the market value of debt when the continuation value for the borrower equals autarky is strictly positive. In the implementation, there will be defaults when the value for the borrower equals autarky. This property implies that the value of debt must be positive during a default. This will in turn require that I allow for partial repayment of debt.

4.4 Optimality of Ex Post Inefficiencies

I now turn to the main result of this section: an efficient allocation calls for ex post inefficient outcomes with strictly positive probability, provided that a sufficient condition is satisfied.

Figure 3: Pareto and utility possibility frontiers.



Region with Ex Post Inefficiencies The next proposition establishes that the utility possibility frontier is upward sloping for borrower values close to autarky.

Lemma 1 *There exists a $\tilde{v} \in (v_a, v^*)$ such that the utility possibility frontier $B(v)$ is strictly increasing over $[v_a, \tilde{v})$ and decreasing over $[\tilde{v}, \bar{v}]$.*

I refer to the interval $[v_a, \tilde{v})$ as the *region with ex post inefficiencies* because for all $v \in [v_a, \tilde{v})$, the market value of debt (and hence the value for the lenders) can be increased by providing higher utility to the borrower, thus making both existing lenders and the borrower better off. This is because supporting a continuation value for the borrower close to the autarkic level requires that a very low level of intermediate goods be employed in production so that the sustainability constraint (8) is satisfied. This depresses production and, consequently, the repayments that the lenders can receive in the period.

In particular, when the borrower's value is close to autarky, intermediates are close to zero (see Proposition 1 part i)). Thus, because of the Inada condition on f, the marginal return from additional intermediates is large enough that the benefit from the extra production which can be obtained by increasing the borrower's continuation value is larger than the cost to the lenders of providing the additional value to the borrower. Therefore, both agents can be made better off relative to autarky, and B is upward sloping in a neighborhood of v_a. In contrast, for sufficiently high promised values, $v \geqslant v^*$, the statically efficient level of intermediates can be supported. For such promised values, increasing the borrower's value is costly and has no benefit for the lenders, and so B is strictly decreasing for $v \geqslant v^*$. Therefore, because of the concavity of B, the utility possibility frontier must peak at some $\tilde{v} \in (v_a, v^*)$. Over the interval $[\tilde{v}, \bar{v}]$, which I will refer to as the *efficient region*, B is decreasing. These results are illustrated in Figure 3.

The Efficient Allocation Transits to the Region with Ex Post Inefficiencies Any efficient allocation starts in the efficient region, because the participation constraint, (PC), in the programming problem (J) can hold as an inequality. For any borrower value, v, in the region with ex-post inefficiencies, (PC) does not bind and $J(v) = J(\tilde{v}) = B(\tilde{v}) > B(v)$. It is optimal for the lenders to promise at least $\tilde{v}$ to the borrower. Instead, for v in the efficient region (PC) in (J) binds and $J(v) = B(v)$. The question now is, Does an efficient allocation transit to the region with ex post inefficiencies after some history, or is the efficient region an ergodic set? Provided that a *sufficient* condition is satisfied, the continuation of any efficient allocation transits to the region with ex post inefficiencies after a sufficiently long (but finite) string of realizations of z_L.

The essential piece of the argument is to show that following a realization of z_L, the continuation utility is strictly lower than the current one: $v'(v, z_L) < v$. To this end, I make the following assumption:

Assumption 3 *The parameters z_L, z_H, μ_H, and γ satisfy the following conditions:*[9]

$$\mu_H z_L^{1-\gamma} \geqslant \mathbb{E}(z^{1-\gamma}), \tag{13}$$

$$(1-\mu_H)\left(\frac{z_L^{1-\gamma}}{\mathbb{E}(z^{1-\gamma})} - 1\right) + \mu_H \left(\frac{z_H}{z_L}\right)^{1-\gamma} \geqslant 1. \tag{14}$$

I will discuss condition (13) below. The technical condition (14) guarantees that the marginal cost of providing continuation utility, $-B'(v)$, is sufficiently high. Under this additional assumption, I can prove the following lemma:

Lemma 2 *Under Assumptions 1 - 3, for every v in the efficient region, $v'_H(v) < v$.*

Lemma 2 is not obvious, because there is a tension between two countervailing forces. First, there is an *incentive effect* that calls for lowering $v'(v, z_L)$ below v. This is because lowering the continuation utility after a low productivity shock helps to separate types and to provide more current consumption when the marginal utility of consumption is

[9] To understand the plausibility of condition (13), consider the following back of the envelope calculation.If z_L is a big recession in which productivity falls by say 10% (relative to productivity in good time $z_H = 1$), and it is fairly unlikely in that the probability of such recession is 5% (so $\mu_H = .95$). Then we have that 3 is satisfied if γ is greater than 1.6.

high. Second, there is a countervailing *commitment effect*: Lowering the continuation utility tightens future sustainability constraints. As is standard in economies with only lack of commitment, there is a motive to backload payments to the sovereign borrower in order to relax future sustainability constraints and allow for lower production distortions in the future. Under Assumption 3, the incentive effect outweighs the commitment effect.

To understand why, consider a necessary first-order condition from the problem

$$B'(v) = \frac{q}{\beta}\left[\mu_L B'(v'(v,z_L)) + \mu_H B'(v'(v,z_H))\right] + \frac{f'(m(v)) - 1}{U'(z_L f(m(v))) f'(m(v))}, \tag{15}$$

which, using the fact that $B'(v) \leqslant 0$ for $v \geqslant \tilde{v}$ and $\beta \leqslant q$, can be rewritten as

$$\left[B'(v'(v,z_L)) - B'(v)\right] \geqslant \mu_H \left[B'(v'(v,z_L)) - B'(v'(v,z_H))\right] - \frac{\beta}{q}\frac{f'(m(v)) - 1}{U'(z_L f(m(v))) f'(m(v))}. \tag{16}$$

Equation (16) illustrates the two forces operating in the model. The first term in square brackets on the right-hand side of (16) stands in for the incentive effect, while the second term stands in for the commitment effect. First notice that by the concavity of B, if the right-hand side of (16) is positive, then it must be that $v'(v,z_L) < v$. By Proposition 1, Part ii), $v'(v,z_H) > v'(v,z_L)$ and, thus, the first term on the right-hand side of (16) is strictly positive. Absent any commitment problem (i.e., $f'(m) = 1$) the second term on the right-hand side of (16) is equal to zero. Therefore, the right-hand side is positive and, consequently, it will be true that $v'(v,z_L) < v$. When the sustainability constraint binds (i.e., $f'(m) > 1$), the second term on the right-hand side of (16), $-[f'(m) - 1] / [U'(z_L f(m)) f'(m)]$, is negative; it is then not obvious that the right-hand side of (16) is positive. Thus, in this case, it is not guaranteed that $v'(v,z_L) < v$. Assumption 3 guarantees that this is indeed the case. Note that such assumption is met either if, for a given μ_H, $z_L^{1-\gamma} - z_H^{1-\gamma}$ is sufficiently large or, for given $z_L^{1-\gamma} - z_H^{1-\gamma}$, μ_L is sufficiently small.

Intuitively, if $z_L^{1-\gamma} - z_H^{1-\gamma}$ is sufficiently large, the benefit of separating the two types is large. It is very cheap to satisfy the promise-keeping constraint by providing consumption when the productivity shock is low, $z = z_L$. To provide a large spread in current consumption across types in an incentive-compatible way, it is necessary to have a large spread in continuation values, $v'(v,z_H) - v'(v,z_L)$. Thus, the first term of the right-hand side of (16) is large. Moreover, if μ_L is small, the cost of tightening future sustainability constraints by reducing the continuation value after z_L is small, from an ex ante perspective. Inspecting (16), if μ_H is low, then the first term on the right-hand side is again large. Thus, if either $z_L^{1-\gamma} - z_H^{1-\gamma}$ is sufficiently large or μ_L is sufficiently small, the benefits from lowering $v'(v,z_L)$ below v by relaxing the incentive compatibility constraint and the current sustainability constraint (incentive effect) are larger than the costs arising from higher production distortions in the future after a high taste shock (commitment effect).

Under the assumptions in Lemma 2, for all ν in the efficient region, $\nu'(\nu, z_L)$ lies *strictly* below the 45 degree line, as illustrated in Figure 4. Let $\Delta \equiv \min_{\nu \in [\tilde{\nu}, \bar{\nu}]} \{\nu - \nu'(\nu, z_L)\}$. By continuity of $\nu'(\nu, z_L)$, it follows that $\Delta > 0$. Thus, starting from any $\nu_0 \in [\tilde{\nu}, \bar{\nu})$, after a sequence of t consecutive realizations of z_L, the borrower's continuation value is less than $\nu_0 - \Delta t$. Thus, after a sufficiently long string $z^T = (z_L, z_L, ..., z_L)$ with $T \leqslant (\nu_0 - \tilde{\nu})/\Delta$, the continuation utility transits to the region with ex post inefficiencies, $[\nu_a, \tilde{\nu})$. The next proposition summarizes the argument above.

Proposition 2 *Under Assumptions 1–3, an ex ante efficient allocation transits to the region with ex post inefficiencies with strictly positive probability.*

Role of the Main Ingredients The interaction between lack of commitment and private information is key to having ex post inefficient outcomes happening along the path in this production economy. Both lack of commitment and the fact that intermediates are used in production are crucial to generating an upward sloping portion of the utility possibility frontier. However, these two features alone cannot generate ex post inefficient outcomes associated with an ex ante efficient allocation. Without an incentive problem, any continuation of an efficient allocation is itself efficient. Thus, an efficient allocation never transits to the region with ex post inefficiencies of the utility possibility frontier. See Aguiar et al. (2009) for this result in a related environment.

Private information alone generates a downward drift of the continuation utility (see Thomas and Worrall (1990) and Atkeson and Lucas (1992)) but does not generate ex post inefficiencies, because with commitment, there is no connection between low continuation values and production in the economy. The statically efficient amount of production can always be sustained. Low continuation values for the borrower only have distributional effects in that the lenders can appropriate larger shares of total undistorted production. Also in this case, continuations of efficient allocations are always on the Pareto frontier.

Both contracting frictions are needed to obtain ex post inefficient outcomes as part of the ex ante optimal arrangement (Proposition 2). Lack of commitment and production are crucial for having an upward sloping portion of the utility possibility frontier (Lemma 1); private information is crucial for having the efficient allocation to transit to the region with ex post inefficiencies (Lemma 2). These features are also present in previous works that also establish the optimality of ex post inefficiencies in related environments, such as Phelan and Townsend (1991), Quadrini (2004), Clementi and Hopenhayn (2006), DeMarzo and Sannikov (2006), and DeMarzo and Fishman (2007).

4.5 Long-run Properties

The next proposition establishes that the efficient allocation features perpetual cycles that transit in and out of the region with ex post inefficiencies.

Proposition 3 *Under Assumptions 1–3, then any efficient allocation converges to a unique non-degenerate stationary distribution. Moreover, the inefficient region, $[v_a, \tilde{v}]$, is in the support of such distribution.*

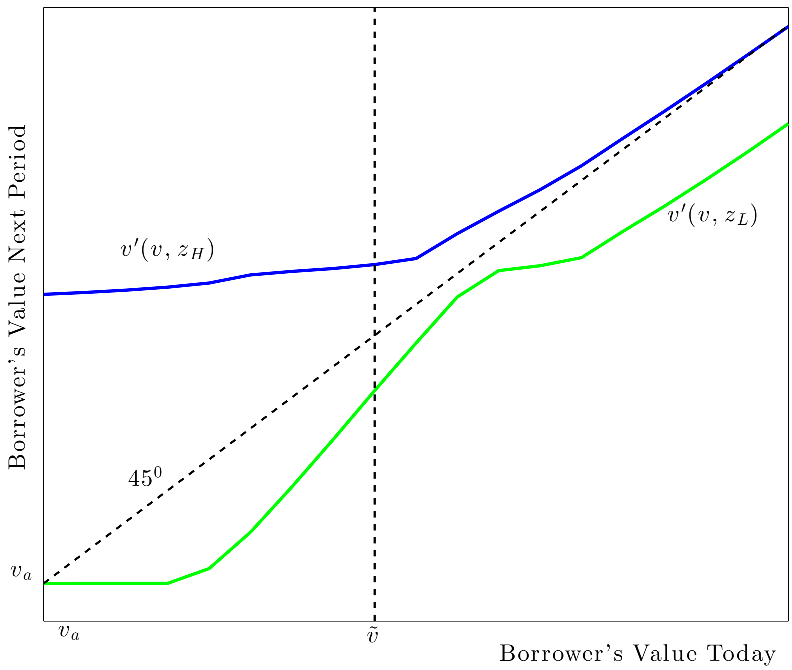
The proof is relegated to the Appendix. The key to understand Proposition 3 is to understand the law of motion for continuation utility illustrated in Figure 4. These laws of motion define a unique ergodic set for promised utility. By Lemma 2, after a sufficiently long—and finite—string of draws of z_L, continuation utility transits to the region with ex post inefficiencies. The region with ex post inefficiencies, and the value of autarky in particular, is not an absorbing state. In the Appendix, I show that whenever z_H is drawn, then the continuation is back in the efficient region. To show that autarky is not an absorbing state, a sufficient condition is that $\beta > q z_H^{1-\gamma} / \mathbb{E}\left(z^{1-\gamma}\right)$. This condition ensures that the government is sufficiently patient so that in the high productivity state it prefers to reduce its consumption below the autarky level, being rewarded with an increase in future consumption. Thus, under these conditions, there is sufficient "mixing" that the existence of a unique limiting distribution is guaranteed.

The limiting distribution has perpetual cycles that transit in and out of the region with ex post inefficiencies. This feature differentiates my environment from related dynamic contracting problems such as those in Quadrini (2004), Clementi and Hopenhayn (2006), DeMarzo and Sannikov (2006), DeMarzo and Fishman (2007), and Hopenhayn and Werning (2008), which also have ex post inefficiencies along the path. In all of these papers, in the long run, either the incentive problem disappears or there is an inefficient termination of the venture between the principal and the agent. In contrast, here the incentive problem does not disappear in the long run, and there is no termination of the risk-sharing relationship. The optimal allocation has periods of temporary autarky, but cooperation eventually restarts after the domestic economy recovers.[10]

The efficient allocation only pins down the transfers between the sovereign borrower and the foreign lenders. It is silent about bond prices, defaults, and so forth. In the next section, I show that the efficient allocation can be implemented given a set of assets typically considered in the quantitative sovereign default literature.

[10]This is because the sovereign borrower is the owner of the domestic production technology that can be also operated in autarky.

Figure 4: Law of motion for borrower's value.



5 Sovereign Debt Game

In this section, I describe the game that implements the efficient allocation. The set of securities that the sovereign borrower can issue is restricted to the ones considered in the literature on quantitative incomplete markets. I impose a set of rules that describe the treatment of the government in default. These rules stand in for the outcome of a renegotiation process and determine the payoff of debt in default. I take these rules as a feature of the environment, and I derive implications for the path of debt and bond prices that implement the efficient allocation given these rules. The fact that the constrained efficient allocation can be implemented implies that this set of rules is not restrictive. In section 6, I discuss how the assumption about this process can be relaxed without affecting the main characteristics of equilibrium outcomes.

Consider a game between competitive foreign lenders (bondholders and exporters of the intermediate good), competitive domestic firms, and a benevolent domestic government. The government can issue two types of non-contingent defaultable bonds: *a one-period bond*, b_S, (or foreign reserves if $b_S < 0$) and a *consol*, $b_L \geqslant 0$. One unit of the one-period bond promises to pay one unit of the export good tomorrow in exchange for q_S units of the export good today. The consol is a perpetuity that promises to pay a coupon of one unit of the export good in every period starting tomorrow in exchange for q_L units of the export good today.

The government cannot commit to satisfying the terms of the bond contracts, and there are *no seniority* clauses; so, in case of default, holders of the consol are treated equally irrespective of the time the consol was issued.[11] The borrower can choose among three options regarding the repayment of inherited debt, $\delta_t \in \{\texttt{full}, \texttt{partial}, \texttt{suspend}\}$.

When $\delta_t =$`full`, there is *full repayment*: the government pays in full the outstanding one-period bond and the coupon on the consol. When $\delta_t =$`partial`, there is *partial repayment*: the government pays a fraction $r_S \in [0, 1)$ of outstanding obligations on one-period bonds and $r_L/(1-q)$ for each unit of consols outstanding with $r_L \in [0, 1)$. Finally, $\delta_t =$`full` denotes *suspension of payments:* the government makes no payments in the current period. It is then excluded from international credit markets, and the defaulted debt is rolled over to the next period (missed interest payments are forgiven) when the government can choose a repayment policy δ_{t+1} for the notional amount of today's debt obligations. I would say that the government is in *default* whenever it repays less than what is contractually specified, that is, $\delta_t \neq$`full`.

I introduce partial repayment as possible choice for the government to decentralize the efficient allocation. This is because the total value of debt in the efficient allocation (that is, the present discounted value of net transfers to foreign lenders) is positive when the value for the borrower equals autarky. I will discuss this issue further in the proof of Proposition 4.

In addition to issuing debt, the government can also tax the payments made by domestic firms to foreign exporters for the intermediate goods at a rate $\tau_t \in [0, 1]$. Thus, foreign exporters receive an after-tax payment of $p_t(1-\tau_t)$ per unit of intermediate good sold, where p_t is the price of the intermediates in terms of the final good. We should think of this as the government interfering with private transactions.[12]

The sequence of events within the period is as follows:

1. Foreign lenders set a price for intermediate inputs p_t (how many units of export good for one unit of intermediate good).

2. Domestic competitive firms choose the quantity of intermediate inputs they want to use, m_t.

3. The productivity shock z_t is realized and privately observed by the domestic government.

4. The government picks a policy $\pi_t = (\delta_t, \mathbf{b}_{t+1}, \tau_t)$ that consists of a repayment decision δ_t, new bond holdings, $\mathbf{b}_{t+1} = (b_{S,t+1}, b_{L,t+1})$, and a tariff on imported inter-

[11] This is consistent with the pari passu clause present in the quasi-totality of sovereign debt contracts.
[12] See Arellano et al. (2015) for evidence of this interference.

mediates, τ_t. Bond holdings are bounded by a large positive constant $\bar{B}$ to rule out a Ponzi scheme.

5. Bond prices $\mathbf{q}_t = (q_{S,t}, q_{L,t})$ are consistent with foreign lenders' optimality.

Following Chari and Kehoe (1990), to formally define a *sustainable equilibrium*, let $h^t = (h^{t-1}, p_t, m_t, \pi_t)$ be a public history up to period $t \geqslant 0$, and let $h^{-1} = \mathbf{b}_0 = (b_{S,0}, b_{L,0})$ be the initial outstanding debt. It is convenient to define the following public histories when agents take action: $h_p^t = h^{t-1}$, $h_m^t = (h^{t-1}, p_t)$, and $h_\sigma^t = (h^{t-1}, p_t, m_t)$. The price of the intermediate good, $p = \{p_t\}_{t=0}^{\infty}$, the allocation rule for $m = \{m_t\}_{t=0}^{\infty}$, the strategy for the government, $\sigma = \{\sigma_t\}_{t=0}^{\infty}$, and the price of bonds, $\mathbf{q} = \{q_{S,t}, q_{L,t}\}_{t=0}^{\infty}$, are all functions of the relevant histories.

Problem of the Government To set up the problem for the government, let $Y(\tau, z) = zF(m_t, 1) - p_t m_t + \tau p_t m_t$ be the amount of resources available to the government after production, repayments of intermediates, and collection of the tariff revenue. Taking as given p, m, and the price schedule for bonds, $\mathbf{q}$, after any history $\left(h_\sigma^t, z\right)$, the strategy for the government, σ, solves the problem

$$\text{(W)} \qquad W(h_\sigma^t, z) = \max_{c, \pi = \left(\delta, b_S', b_L', \tau\right)} U(c) + \beta \mathbb{E}\left[W(h_\sigma^{t+1}, z_{t+1}) | h_\sigma^t, \pi\right]$$

subject to $b_S', b_L' \leqslant \bar{B}$ and the consolidated budget constraints of the government and the stand-in household.[13] If there is no default (i.e., if $\delta =$`full`), the consolidated budget constraint is given by

$$\text{(17)} \qquad c + (b_{S,t} + b_{L,t}) \leqslant Y(\tau, z) + q_{S,t}(h_\sigma^t, \pi) b_S' + q_{L,t}(h_\sigma^t, \pi)(b_L' - b_{L,t}),$$

or, if there is partial repayment (i.e., if $\delta =$`partial`) by

$$\text{(18)} \qquad c + \left(r_S b_{S,t} + r_L \frac{b_{L,t}}{1-q}\right) r \leqslant Y(\tau, z) + q_{S,t}(h_\sigma^t, \pi) b_S' + q_{L,t}(h_\sigma^t, \pi) b_L',$$

or, if there is default without any partial repayment (i.e., if $\delta =$`suspend`), then I impose the restriction that after $\delta_t =$`suspend`, there is a temporary exclusion from international credit markets. That is,

$$\text{(19)} \qquad c \leqslant Y(\tau, z) \quad \text{and} \quad \left(b_S', b_L'\right) = (b_{S,t}, b_{L,t}).$$

[13] In the background, as in Aguiar et al. (2009), the stand-in domestic household supplies labor inelastically and receives lump sum transfers (or taxes if negative), LS_t, from the government. The stand-in household's budget constraint is $c_t = w_t + LS_t$, where $w_t = F_\ell(m_t, 1)$ is the competitive wage rate. (17)-(19) represent the combined budget constraints of the benevolent government and the stand-in household.

Bond Prices and Other Equilibrium Objects The price of the imported intermediate, p_t, must be consistent with optimization by competitive foreign lenders who take the tariff level as given after all histories:

$$1 = \mathbb{E}\left[p_t(h_p^t)\left(1-\tau_t(h_\sigma^t, z_t)\right)|h_p^t\right]. \tag{20}$$

The allocation rule for the quantity of foreign intermediate goods, m_t, satisfies the optimality condition for the representative domestic competitive firm after all histories:

$$\mathbb{E}\left[Q(h_\sigma^t, z_t)F_m(m_t(h_m^t), 1)|h_m^t\right] = p_t(h_p^t), \tag{21}$$

where $Q(h_\sigma^t, z_t) = U'\left(c(h_\sigma^t, z_t)\right)/\mathbb{E}\left[U'\left(c(h_\sigma^t, z_t)\right)\right]$ is the price that the representative domestic firm uses to evaluate profits.

Given the government repayment policy, bond prices $q_{S,t}, q_{L,t}$ are consistent with the maximization problem of the risk-neutral foreign lenders who discount the future at a rate q. For the one-period bond, if $b_{S,t+1} \geqslant 0$, it must be that

$$q_{S,t}(h^t) = q\mathbb{E}\left[\chi_{S,t+1}(h^{t+1})|h^t\right], \tag{qS}$$

where $\chi_{S,t+1}$ is the ex post value of short-term debt:

$$\chi_{S,t+1}(h^{t+1}) = \begin{cases} 1 & \text{if } \delta_{t+1}(h_\sigma^{t+1}) = \texttt{full} \\ r_S & \text{if } \delta_{t+1}(h_\sigma^{t+1}) = \texttt{partial} \\ q\mathbb{E}\left[\chi_{S,t+2}(h^{t+2})|h^{t+1}\right] & \text{if } \delta_{t+1}(h_\sigma^{t+1}) = \texttt{suspend} \end{cases} \tag{22}$$

Here the expectation in (qS) is taken at the end of period t over next period's value of the productivity shock, z_{t+1}. The lenders understand the repayment rule of the government, and the price is simply the actuarially fair value of repayments. The only subtle part is that if the government does not repay the short bond at $t+1$ by setting $\delta_{t+1}(h_\sigma^{t+1})$ $=$`suspend`, this bond still has value because it gives the holder the right to any partial repayment r_S that the government may make on this defaulted debt either in period $t+2$ or some later period in order to regain access to the bond market. In this sense, when δ_{t+1} $=$`suspend`, $q\mathbb{E}\left[\chi_{S,t+2}(h^{t+2})|h^{t+1}\right]$ represents the secondary market value of defaulted debt.

Next, consider situations in which the government saves, in that $b_{S,t+1} < 0$. If the history is such that the government is not excluded from saving, the rate on foreign assets will equal the world rate, that is, $q_{S,t}(h^t) = q$. If instead the history is such that the government is being temporarily excluded from the international credit market, I adopt the convention that $q_{S,t}(h^t) = \infty$. Allowing for exclusion from savings is necessary to decentralize the efficient risk-sharing arrangement. In such arrangement, detectable deviations are punished with permanent autarky. If there is no mechanism preventing the government to save after a detectable deviation, then the government could default and self-insure via saving. This would increase the incentive for the government to deviate and prevent to decentralize the efficient allocation with this market structure.

Finally, the price for the consol must be such that

$$q_{L,t}(h^t) = q\mathbb{E}\left[\chi_{L,t+1}(h^{t+1})|h^t\right], \tag{qL}$$

where $\chi_{L,t+1}$ is the ex post value of the consol given by

$$\chi_{L,t+1}(h^{t+1}) = \begin{cases} 1 + q_{L,t+1}(h^{t+1}) & \text{if } \delta_{t+1}(h_\sigma^{t+1}) = \texttt{full} \\ \frac{r_L}{1-q} & \text{if } \delta_{t+1}(h_\sigma^{t+1}) = \texttt{partial} \\ q\mathbb{E}\left[\chi_{L,t+2}(h^{t+2})|h^{t+1}\right] & \text{if } \delta_{t+1}(h_\sigma^{t+1}) = \texttt{suspend} \end{cases} \tag{23}$$

If the government does not repay its consol at $t+1$, this consol still has value because it gives the holder the right to any partial repayment $r_L/(1-q)$ that the government may make in future periods in order to regain access to the bond markets.

Equilibrium Definition Given initial outstanding debt $\mathbf{b}_0$, a *sustainable equilibrium* is a strategy for the government, σ, a price rule for the foreign intermediate good, p, price rules for the government bonds, q_S and q_L, and an allocation rule for the intermediate good, m, such that i) given p, m, q_S, and q_L, the government's strategy σ, is the optimal policy associated with (W) for all (h_σ^t, z); ii) given σ, the price and allocation rules p and m satisfy (20) and (21); iii) given σ, the price of the short-term bond q_S satisfies (qS) whenever $b_S \geqslant 0$, and it equals q or ∞ when $b_S < 0$, where $q_S = \infty$ stands in for exclusion from saving abroad; iv) given σ, the price of the long-term bond q_L satisfies (qL).

The associated *equilibrium outcome* is denoted by $\mathbf{y} = (\mathbf{x}, \mathbf{g}, \mathbf{p})$, where $\mathbf{x} = \left\{m(z^{t-1}), c(z^t)\right\}_{t=0}^{\infty}$, $\mathbf{g} = \left\{\delta(z^t), b_L(z^t), b_S(z^t), \tau(z^t)\right\}_{t=0}^{\infty}$, and $\mathbf{p} = \left\{p(z^{t-1}), \mathbf{q}(z^t)\right\}_{t=0}^{\infty}$.

It is useful to characterize the set of outcomes that can be implemented as a sustainable equilibrium of the sovereign debt game. Such characterization gives us a set of necessary and sufficient conditions for an equilibrium outcome, which I will later show that are met by the efficient allocation.

The logic of the characterization follows Abreu (1988) and Chari and Kehoe (1990) in using reversion to the worst equilibrium to characterize the set of equilibrium outcomes. The worst equilibrium outcome from the borrower's perspective is the autarkic allocation. The autarkic allocation can be supported as an equilibrium outcome as follows: Zero intermediates and a price equal to zero for both short-term and long-term debt can be supported as part of a sustainable equilibrium, because if foreign lenders expect a tariff equal to 100% (full expropriation) and full default in any subsequent periods irrespective of the action chosen today by the government ($\delta_t =$`suspend` for all subsequent t), then the government has no incentive to choose something different than $\tau_t = 1$ and $\delta_t =$`suspend`, confirming the lenders' beliefs. The fact that the sovereign borrower cannot save after a deviation follows from an assumption commonly used in the literature (e.g., Atkeson (1991) and Aguiar et al. (2009)) to rule out the classic Bulow and Rogoff (1988) result that no foreign debt can be sustained in equilibrium.

The next Lemma characterizes the set of sustainable equilibrium outcomes:

Lemma 3 *Given initial outstanding debt* $\mathbf{b}_0$*,* $\mathbf{y}$ *is a* sustainable equilibrium outcome *if and only if it satisfies the resource constraint (RC), the incentive compatibility constraint (IC), and the sustainability constraint, (SUST) in addition to the budget constraints (17)–(19) and (20), (21), (qS) and (qL) along the equilibrium outcome path.*

Proof. The fact that the conditions in Lemma 3 are necessary for an equilibrium outcome is evident. Conditions (20), (21), (qS), and (qL) are part of the definition of a sustainable equilibrium. Budget feasibility from the government's perspective requires the outcome to satisfy (17)–(19). Constraints (IC) and (SUST) ensure that the sovereign borrower has no strict incentive to engage in a detectable and undetectable deviation, respectively. In (SUST), I use the fact that the borrower's value after a detectable deviation cannot be lower than the value of the worst equilibrium, v_a.

To see that such conditions are also sufficient, consider an outcome that satisfies the conditions in the statement. Such outcome can be implemented as sustainable equilibrium by relying on trigger strategies that revert to the worst equilibrium after a detectable deviation from the government. Clearly, such an outcome satisfies the optimality conditions for the competitive agents, since it satisfies (20), (21), (qS), and (qL), and it is budget feasible for the government, since it satisfies the budget constraints (17)–(19). Then I am left to show that the government has no incentive to deviate from the prescribed path of plays. To this end, note the detectable deviations are not profitable, because the outcome satisfies the sustainability constraint (SUST). Moreover, undetectable deviations are not profitable, because the outcome satisfies the incentive compatibility constraint (IC). Then, the government does not have a strict incentive to deviate from the proposed path of plays, concluding the proof. Q.E.D.

6 Defaults, Bond Prices, and Maturity Composition

I now show that any efficient allocation can be implemented as an equilibrium outcome of the sovereign debt game. This amounts to say that the best equilibrium outcome of the sovereign debt game is (constrained) efficient. Along the equilibrium outcome path, there are defaults only when the borrower's continuation value is equal to the autarkic value. Defaults along the path do not trigger permanent exclusion from international credit markets. In the model, the borrower suspends payments to foreign lenders until a high productivity shock is drawn and a given partial repayment on the defaulted debt is made. After such partial repayment, the borrower regains access to foreign borrowing and lending.

It is important to note the distinction between a detectable deviation from the path of play, which may include a default, and what happens along the equilibrium path when there is a default. On-path defaults are *excusable* in the sense introduced by Grossman and Van Huyck (1988): They happen in well-understood circumstances, and the country regains access to international credit markets after making a partial repayment; therefore, on-path defaults do not trigger autarky forever.

Implementation In the rest of the paper, I will assume that the efficient allocation satisfies the following properties:

Assumption 4 *The efficient allocation is such that*

i) for all $v > v_a$, $v'(v, z_L) < v$ and there exists a $\underline{v} \in (v_a, \bar{v})$ such that $v'(v, z_L) = v_a$ for all $v \leqslant \underline{v}$;

ii) $v'(\cdot, z)$ is strictly increasing for all $v \geqslant \underline{v}$ and for all z.

Part i) implies that starting from any v, there is a strictly positive probability of reaching autarky after a sufficiently long but finite string of low productivity shocks. This property does not follow from Lemma, 2 since the Lemma does not immediately extend to the region of ex-post inefficiencies if lenders are more impatient than the borrower. This is because for v such that $B'(v) > 0$, it is more costly to front-load utility for the relatively more patient lenders (the opposite of what happens for promised utility v such that $B'(v) \leqslant 0$). In the Appendix, I provide two sets of sufficient conditions on primitives that ensure the efficient allocation satisfies property i). In particular, I show that if $f(m^*) - f(0)$ is sufficiently small or if β is sufficiently close to q, then the condition is satisfied. Moreover, if the efficient allocation displays partial insurance, in that $U'(c_L) \geqslant U'(c_L)$, then a stronger version of condition (13) ensures that part i) holds.[14] Part ii) requires that the continuation values, $v'(\cdot, \theta_L)$ and $v'(\cdot, \theta_L)$, be increasing in current promised utility. Both i) and ii) are satisfied in my simulations (e.g., Figure 4).

[14]If the property in part i of Assumption 4 is not satisfied by the efficient allocation, it is still possible to implement the efficient allocation but it requires defaults for $v \neq v_a$.

For later reference, define the *short-term spread* as the difference between the interest rate implied by q_S and the risk-free international interest rate: $s_S \equiv 1/q_S - 1/q$. The *long-term spread* is defined as the difference between the consol's yield to maturity[15] and the risk-free interest rate: $s_L \equiv (1+q_L)/q_L - 1/q$. The *term premium* is the difference between the long- and the short-term spreads: $s_T \equiv s_L - s_S$.

The main result of this section is that there exists a sustainable equilibrium that decentralizes the efficient allocation such that default happens along the equilibrium path in the region of ex post inefficiencies when the government's debt is high (relative to the maximal amount that can be sustained) and output is low. Furthermore, if the recovery rate on long-term bonds is sufficiently high relative to the recovery rate on short-term bonds,[16] the equilibrium outcome is such that when spreads are low, the term premium is positive, but it delivers an inversion of the yield curve when spreads are high. Moreover, numerical simulations show that the maturity composition of debt shortens when spreads are high.

Proposition 4 *Under Assumptions 1–3 and $\beta < q$, given an efficient allocation that satisfies Assumption 4 and a set of recovery rates (r_S, r_L) with at least one rate greater than zero, a sustainable equilibrium exists that decentralizes it, with default happening along the equilibrium path when the continuation value for the borrower equals v_a. Moreover, if $r_L \geqslant r_S$, the equilibrium is such that the term premium is positive when borrower's value is above v_a and negative at v_a.*

The proof of the proposition is by construction. I construct the on-path default rule, bond holdings, tariffs, and prices that support the efficient allocation and are consistent with the sufficient condition for a sustainable equilibrium outcome in Lemma 3. Since the efficient allocation can be represented by a time-invariant function of borrower's continuation utility and exogenous shocks, the on-path repayment rule, bond holdings, tariffs, and prices can also be expressed as a function of on-path continuation utility for the borrower and the current realization of the productivity shock z:

$$\bar{\tau}, \bar{p} \; : \; [v_a, \bar{v}] \to \mathbb{R}, \; \bar{\delta} : [v_a, \bar{v}] \times \{z_L, z_H\} \to \{\texttt{full}, \texttt{partial}, \texttt{suspend}\} \tag{24}$$

$$\bar{q}_S, \bar{q}_L, \bar{b}_S, \bar{b}_L \; : \; [v_a, \bar{v}] \times \{z_L, z_H\} \to \mathbb{R} \tag{25}$$

Note that I consider a decentralization in which $\bar{\tau}$ does not depend on z to emphasize how active maturity management can replicate state contingent return implicit in the efficient allocation. A state contingent tariff can also provide insurance to the government; for

[15]That is the implicit constant interest rate at which the discounted value of the bond's coupons equals its price. Define $q_{YM,L}$ as $q_L = \frac{q_{YM,L}}{1-q_{YM,L}}$. The consol's yield to maturity is $1/q_{YM,L} = \frac{q_L}{1+q_L}$.

[16]Recent works have documented that this is actually true in the data. See for example Zettelmeyer et al. (2013) and Asonuma and Ranciere (2015).

instance, see the role of state contingent taxes on capital income in Aguiar et al. (2009).[17]

An outcome path $\mathbf{y}$ can be recovered in the natural way from (24), (25), and the law of motion for v from the efficient allocation. Moreover, bond holdings and prices depend only on the continuation value after z is realized, $v'(v,z)$. With some abuse of notation, I can then write:

$$\bar{q}_S(v,z) = \bar{q}_S(v'(v,z)),\ \bar{q}_L(v,z) = \bar{q}_L(v'(v,z)), \tag{26}$$

$$\bar{b}_S(v,z) = \bar{b}_S(v'(v,z)),\ \bar{b}_L(v,z) = \bar{b}_L(v'(v,z)). \tag{27}$$

The steps to construct the candidate equilibrium outcome path $\mathbf{y}$ from an efficient allocation $\mathbf{x}$ are as follows: i) define the repayment policy; ii) use the repayment policy in the optimality conditions for the foreign lenders to calculate equilibrium bond prices; iii) choose short- and long-term debt to match the total value of debt (lenders' value) after a realization of z implied by the efficient allocation

$$b(v,z) = f(m(v)) - c(v,z) - m(v) + qB(v'(v,z)), \tag{28}$$

and iv) use the optimality conditions for the domestic firms and the lenders to get tariffs and prices for the intermediate good.

Consider first the repayment policy. The borrower defaults only when his continuation value is equal to the autarky value, v_a. For all other borrower values, $v > v_a$, there is full repayment. In particular,

$$\bar{\delta}(v,z) = \begin{cases} \texttt{suspend} & \text{if } v = v_a \text{ and } z = z_L \\ \texttt{partial} & \text{if } v = v_a \text{ and } z = z_H \\ \texttt{full} & \text{if } v > v_a \text{ for all } z \end{cases}, \tag{29}$$

where $\bar{\delta}(v_a, z_L) = \texttt{suspend}$ because, as showed in Lemma 10 in the Appendix, when the borrower's value is autarky, there are no capital flows in that $m(v_a) = 0$ and $c(v_a, z_L) = f(0)$. When $z = z_H$, $c(v_a, z_H) < f(0)$ and so there is an outflow of resources. This is matched by a partial repayment on defaulted debt.

Given the repayment policy, bond prices are uniquely pinned down by the lenders' optimality conditions. The price for short-term debt is given by

$$\bar{q}_S(v) = \begin{cases} q & \text{if } v > v_a \\ q\bar{R}_S & \text{if } v = v_a \end{cases}, \tag{30}$$

[17] An attractive feature of the decentralization I propose is that it extends to pure exchange economies, while the decentralization based on state contingent tariff or taxes does not.

where $\bar{R}_S$ is the expected repayment in case of default for short-term debt, $\bar{R}_S = \mu(z_H) r_S / (1 - q\mu(z_L))$. The price for long-term debt can be written recursively as

$$\bar{q}_L(v) = \begin{cases} q \sum_{i=L,H} \mu(z_i) \left[1 + \bar{q}_L(v'_i(v))\right] & \text{if } v > v_a \\ \frac{q}{1-q} \bar{R}_L & \text{if } v = v_a \end{cases}, \tag{31}$$

where $\bar{R}_L q / (1-q)$ is the expected repayment in case of default for long-term debt, $\bar{R}_L = \mu(z_H) r_L / (1 - q\mu(z_L))$. For all $v > v_a$, the price of short-term debt is equal to that of a risk-free bond. Instead, the price of long-term debt is lower than the price of a risk-free consol, because there is always a positive probability that there will be a default over the relevant time horizon of the bond.

The next Lemma shows that $\bar{q}_L$ is increasing in the continuation value for the borrower, and it establishes that the price of long-term debt increases after drawing z_H, in that $\bar{q}_L\left(v'_H(v)\right) > \bar{q}_L\left(v'_L(v)\right)$.

Lemma 4 *Under the assumptions in Proposition 4, $\bar{q}_L : [v_a, \bar{v}] \to \mathbb{R}$ is the unique fixed point of the contraction mapping defined by the right-hand side of (31), it is increasing, and for all v we have*

$$\frac{q}{1-q} > \bar{q}_L\left(v'_H(v)\right) > \bar{q}_L\left(v'_L(v)\right).$$

The proof for this Lemma is provided in the Appendix. In the proof, I use the assumption that $v'(v, z)$ is monotone in v for all z, part ii) of Assumption 4.

Given the functions for bond prices $\bar{q}_S$ and $\bar{q}_L$, in the no-default region, $\bar{b}_S(v)$ and $\bar{b}_L(v)$ are chosen to match the total value of debt (lenders' value) implied by the efficient allocation after z_H and z_L defined in (28):

$$b(v, z_H) = \bar{b}_S(v) + \bar{b}_L(v)\left[1 + \bar{q}_L(v'_H(v))\right] \tag{32}$$

$$b(v, z_L) = \bar{b}_S(v) + \bar{b}_L(v)\left[1 + \bar{q}_L(v'_L(v))\right]. \tag{33}$$

A unique solution to (32)–(33) is guaranteed by Lemma 4, which establishes that $\bar{q}_L\left(v'_H(v)\right) > \bar{q}_L\left(v'_L(v)\right)$, meaning that the price of the long-term bond falls after z_L is realized relative to z_H. Thus, outside the default region, the maturity composition of debt is uniquely pinned down. Simple algebra shows that

$$\bar{b}_L(v) = \frac{b(v, z_H) - b(v, z_L)}{\bar{q}_L(v'_H(v)) - \bar{q}_L(v'_L(v))} \tag{34}$$

$$\bar{b}_S(v) = b(v, z_H) - \bar{b}_L(v)\left[1 + \bar{q}_L(v'_H(v))\right]. \tag{35}$$

Notice that it is guaranteed that $\bar{b}_L(v) > 0$, because, by Proposition 1 part iii), $b(v, z_H) - b(v, z_L) > 0$ and $\bar{q}_L(v'_H(v)) - \bar{q}_L(v'_L(v)) > 0$.

When the continuation value for next period onward is equal to autarky, the split between long- and short-term debt is indeterminate; $\bar{b}_S(v_a)$ and $\bar{b}_L(v_a)$ must satisfy

$$b(v_a, z_H) = r_S \bar{b}_S(v_a) + r_L \frac{\bar{b}_L(v_a)}{1-q}. \tag{36}$$

The other possible outcome follows from (36) and $\delta(v_a, z_L) =$suspend because

$$b(v_a, z_L) = q\left[\mu(z_H) b(v_a, z_H) + \mu(z_L) b(v_a, z_L)\right] = \frac{q\mu(z_H)}{1 - q\mu(z_L)} b(v_a, z_H). \tag{37}$$

Thus, for any given recovery rates (r_S, r_L), I can choose any $\left(\bar{b}_S(v_a), \bar{b}_L(v_a)\right)$ that satisfies (36). Consequently, (37) will also be satisfied. Note that to be able to satisfy (36), it is necessary that at least one between r_S and r_L be strictly positive. If both r_S and r_L are strictly positive, the maturity composition is undetermined. I resolve the indeterminacy by assuming that $\bar{b}_S(v_a)/\bar{b}_L(v_a) = \lim_{v \to v_a} \bar{b}_S(v)/\bar{b}_L(v)$. The recovery rates (r_S, r_L) are free parameters. They can be chosen to be sufficiently low so that $\bar{b}_S$ is strictly positive in the default region and for v close to v_a so that a non-full repayment has a natural interpretation.

Finally, on-path tariff rates and prices for the intermediate good, $\bar{\tau}, \bar{p}$, are given by[18]

$$\mathbb{E}\left[\frac{U'\left(c(v,z)\right)}{\mathbb{E}\left[U'\left(c(v,z)\right)\right]} f'(m(v))\right] = \frac{1}{1-\bar{\tau}(v)} = \bar{p}(v). \tag{38}$$

Then the outcome path $\mathbf{y}$ constructed from the efficient allocation $\mathbf{x}$ using (29), (30), (31), (34), (35), (36), and (38) satisfies the optimality conditions (20) and (21), the equilibrium bond pricing equations (qS) and (qL), and the budget constraints (17)–(19) along the equilibrium path. Since the efficient allocation satisfies the incentive compatibility constraint (IC) and the sustainability constraint (SUST), this concludes the proof of the first part of the Proposition, as it verifies that the constructed outcome satisfies the sufficient conditions for a sustainable equilibrium outcome in Lemma 3.

I am left to characterize the behavior of the term structure of interest rates. To this end, first notice that outside the default region, the short-term debt is risk-free; see (30). Thus, $q_{S,t} = q$ and $s_{S,t} = 0$. Instead, from Lemma 4 we know that $q_{L,t} < q/(1-q)$. Therefore, the short-term spread is zero, while the long-term spread is positive. Consequently, the term spread s_T is positive. When the borrower's continuation value is in the default

[18]The necessity of tariff is related to the necessity of capital income taxes in the implementation for the efficient allocation in economy with lack of commitment in Kehoe and Perri (2004) and Aguiar et al. (2009).

region, the term spread is given by

$$s_T(\nu_a) = \left(1 + \frac{1}{\bar{q}_L(\nu_a)}\right) - \frac{1}{\bar{q}_S(\nu_a)} = -\left(\frac{1-\bar{R}r_L}{\bar{R}r_L}\right) - \frac{1}{q\bar{R}}\left(\frac{r_L - r_S}{r_S r_L}\right), \tag{39}$$

which is negative provided that the recovery rate on short-term debt r_S is not too small relative to the recovery rate for long-term debt r_L. A sufficient condition is $r_L \geqslant r_S$. Q.E.D.

Proposition 4 can be generalized to the case in which z can take on more than two values, allowing for a richer maturity structure. For instance, if z can take on N values, then I can use N types of the perpetuity considered in Hatchondo and Martinez (2009) that pay a coupon which decays exponentially at rate $\alpha_n \in [0, 1]$. The one-period bond and the consol are special cases of this class of securities for α equal to 1 and 0, respectively. Provided that the return matrix satisfies a full-rank condition[19] (which is satisfied when $N = 2$ because of Lemma 4), the statement in Proposition 4 generalizes to the case with $N > 2$.

Mechanism that Replicates State-Contingent Returns The crucial step proving that the efficient allocation can be an outcome of the sovereign debt game was to show that it is possible to replicate the insurance provided by the efficient allocation, that is, the total value of debt falls after an adverse shock relative to a positive shock, $b(\nu, z_H) > b(\nu, z_L)$.

How is insurance provided in the sovereign debt game? When there is default, partial repayments make the non-contingent debt de facto state-contingent. When there is full repayment, the fall in the value of debt after the realization of a low productivity shock is obtained by *diluting outstanding long-term debt*, that is, by imposing a capital loss on the holders of outstanding long-term debt. After a low productivity shock, the continuation value for the borrower decreases, the overall level of indebtedness increases, and the probability that there will be a default in the near future increases. This increase in the likelihood of a future default reduces the value of the outstanding long-term debt, resulting in a capital loss for the debt holders and a capital gain for the borrower. This capital

[19] For $\alpha_i \in \{\alpha_1 = 0, \alpha_2, ..., \alpha_N = 1\}$, define $\bar{q}_{\alpha_i}$ in a similar way as in (31):

$$\bar{q}_{\alpha_i}(\nu') = \begin{cases} q \sum_\theta \mu(\theta)\left[1 + (1-\alpha_i)\bar{q}_{\alpha_i}(\nu'(\nu,\theta))\right] & \text{if } \nu' > \nu_a \\ q\frac{\bar{R}_{\alpha_i}}{1-(1-\alpha_i)q} & \text{if } \nu' = \nu_a \end{cases}$$

Then, if the return matrix

$$\underset{N\times N}{\bar{Q}(\nu)} \equiv \begin{bmatrix} 1 + q_{\alpha_1}(\nu'(\nu, z_1)) & ... & 1 + q_{\alpha_N}(\nu'(\nu, z_1)) \\ 1 + q_{\alpha_1}(\nu'(\nu, z_2)) & ... & 1 + q_{\alpha_N}(\nu'(\nu, z_2)) \\ ... & ... & ... \\ 1 + q_{\alpha_1}(\nu'(\nu, z_N)) & ... & 1 + q_{\alpha_N}(\nu'(\nu, z_N)) \end{bmatrix}$$

is invertible, then there exists a $\bar{b}(\nu) = [\bar{b}_{\alpha_1}, ..., \bar{b}_{\alpha_N}]^T$ that solves the analogue of (32)-(33) given $\bar{Q}$.

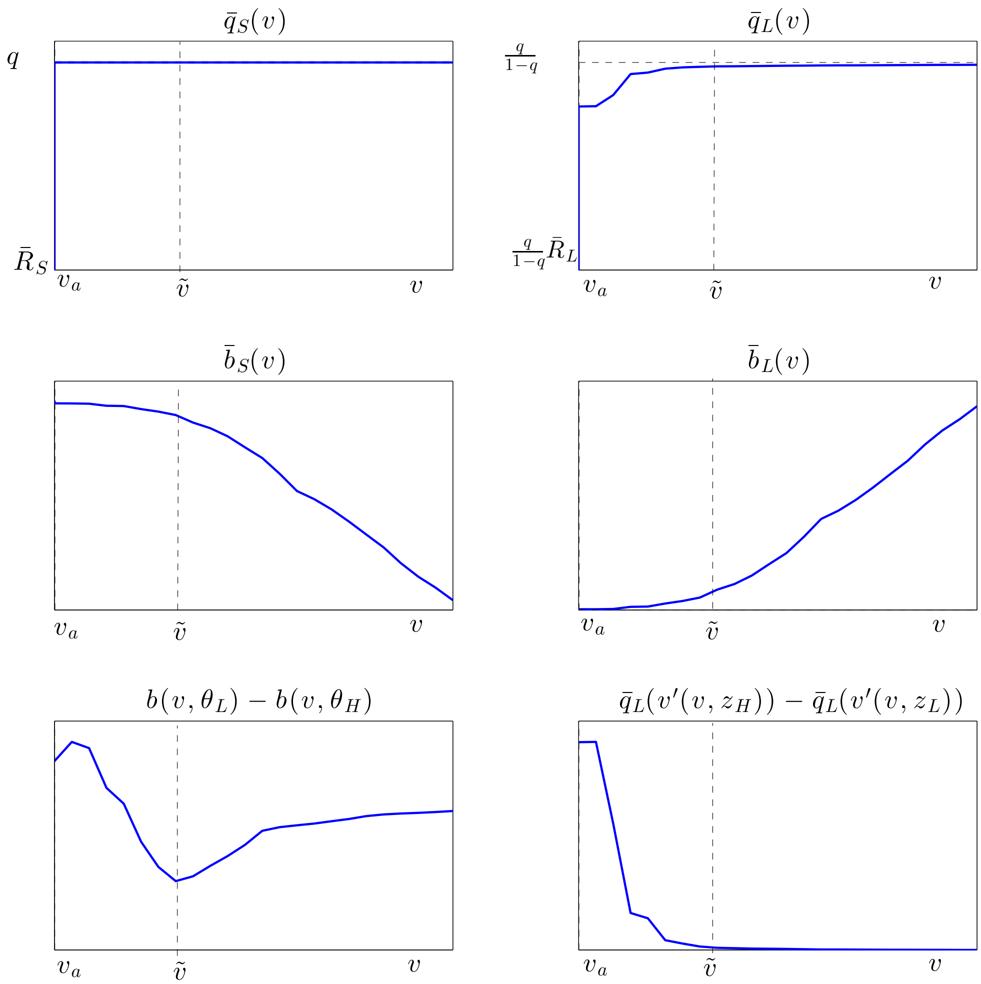
loss on the debt holders after an adverse shock mimics the debt relief for the borrower associated with the efficient allocation. This capital loss in a low productivity shock is compensated by a capital gain after a high productivity shock, so on average the lenders break even. The maturity composition of debt is driven solely by the requirement of matching this differential value of government debt ex post .

Maturity Shortens as Interest Rate Spread Increases I now turn to the implications for the optimal maturity composition of debt. The main finding is that the maturity of outstanding debt issued by the sovereign borrower shortens as the long term spread increases. In particular, the amount of long-term debt decreases, while the amount of short-term debt increases for all v in the efficient region. This result is illustrated in Figure 5. I cannot state a proposition for this result, but the findings are consistent in all of my numerical simulations.

To understand this result, notice that outside the default region, the amount of long-term debt issued by the borrower is determined by (34). Given the ex post variation in the price of the consol, $\bar{q}_L(v'_H(v)) - \bar{q}_L(v'_L(v))$, the long-term debt holdings are constructed to match the debt relief implied by the optimal contract after the realization of z_L, $b(v, z_H) - b(v, z_L)$. As is shown in Figure 5, the level of debt relief is approximately constant for all v over the efficient region. The ex post variation in the price of the consol instead is larger the closer the borrower is to the default region. This is because as the borrower's continuation value approaches the default threshold from above, it is more likely that a realization of a low productivity shock will push the economy into default in the near future. Hence, the long-term debt price is more sensitive to the realization of the shock. Therefore, a lower holding of long-term debt is needed in order to replicate the same amount of insurance, that is, the same debt relief after a high taste shock. Since the overall level of indebtedness is increasing, it must be that $\bar{b}_S$ is increasing as the borrower's continuation value approaches $\tilde{v}$, because $\bar{b}_L$ is falling at the same time. Therefore, in the efficient region, as the level of indebtedness and the spread on long term debt s_L increase, the maturity composition of debt shortens.

In the region with ex post inefficiencies, $[v_a, \tilde{v}]$, the ratio of short-term debt to long-term debt is not always decreasing in the borrower's value under all parameterizations. This is because the ex-post variation in the price of long-term debt is high, but also the amount of insurance, $b(v, z_H) - b(v, z_L)$, increases a lot in this region (see Figure 5). Despite not necessarily being monotonically decreasing in this region, the maturity composition of debt is more tilted toward short-term debt than it is for continuation values associated with lower default probabilities.

Figure 5: Bond prices and holdings, insurance and ex-post variation of the long-term debt price.



Assumptions on Rules in Default I now turn to discuss the conventions I choose for the government in default and whether they can be relaxed without affecting the main characteristics of equilibrium outcomes. To this end, it is important to understand the dynamics of payments prescribed by the efficient allocation. In the Appendix (Lemma 9), I show that when the borrower's value equals the value of autarky, there are no capital flows when z equals z_L, $x(\nu_a, z_L) = 0$, and there are outflows when z equals z_L, $x(\nu_a, z_H) > 0$ and $b(\nu_a, z_H) > 0$ (see also Proposition 1 part iv). The efficient allocation only pins down total payments when the value of the borrower is autarky and it draws a positive productivity shock, as illustrated in equation (36). This implies a degree of indeterminacy at ν_a. In fact, if the recovery rates double and the face value of debt is halved, the borrower makes the same payment and raises the same resources in the previous period as the price of debt doubles from equations (30) and (31). The fact that $b(\nu_a, z_H) > 0$ requires that at least one between r_S and r_L be strictly positive. Other than this, there are no other requirements on recovery rates.

Total payments prescribed by the efficient allocation after the economy recovers do not depend on the length the country spent in temporary autarky, say $n \geqslant 0$. This implies that if the recovery rates r_S and r_L do not depend on n, then the interest rates arrears must be forgiven so that the equilibrium payout received by lenders does not depend on n. Clearly, an alternative way to implement the efficient allocation is to have interest rate arrears not being forgiven and recovery rates that depend on n. This clearly would not change the behavior of the equilibrium outcome leading to a default.

In setting up the sovereign debt game, I assumed that the government settles with the holders of legacy debt by making a current payment. Nothing will change if the government could use a mix of current payments and newly issued debt to pay existing debt holders as part of the settlement agreement, as in Benjamin and Wright (2009). It is worth noticing that some payment must be done in the current period, as the total net export of the country is positive, $x(\nu_a, z_H) > 0$.

Finally, several authors (e.g., Tomz and Wright (2013) and references therein) have argued that market access and interest rates differ depending on the "haircut" applied on defaulted debt and the history of previous default. This is inconsistent with the dynamics of the efficient allocation considered here. Adding an extra dimension of asymmetric information may help in this regard. In particular, one can account for this fact by introducing private information about the type of borrower, which is only revealed by a default (and not by other actions). Although interesting, this is outside the scope of this paper.

To summarize, in this section, I showed that an efficient allocation can be implemented with only non-contingent defaultable debt of multiple maturities. Along the equilibrium outcome path, defaults are associated with an ex post inefficient drop in output and trade, and inversion of the yield curve, and happen only when the borrower's value is equal to autarky and the level of debt is high relative to the maximal amount of debt the country can support. When there is no default, capital gains or losses on outstanding long-term debt replicate the state contingent returns implied by the efficient allocation. Moreover, the maturity of outstanding debt shortens as interest rate spreads increase.

7 Discussion of Implementation

Maturity Management and Insurance Provision A common criticism of using maturity composition to complete markets is that it requires extremely large offsetting positions in long- and short-term debt in plausibly calibrated economies. This is because the variation of the equilibrium stochastic discount factor in response to aggregate shocks is small relative to the amount of insurance in an Arrow-Debreu economy. See the discussion in Buera and Nicolini (2004).

This criticism only partially applies to the environment in this paper for two reasons. First, the amount of insurance is limited by the presence of informational and commitment frictions. Second, variations in the long-term debt price are generated by variations in the default probability, not by variations in the stochastic discount factor of the lenders. When a foreseeable default can happen only in the distant future, shocks have limited impact on the price of defaultable debt. Hence, implausibly large offsetting positions in long- and short-term debt are required to mimic the state contingent return. However, if the probability of default in the near future is sufficiently high and recovery rates are sufficiently low, then the price of long-term debt is very sensitive to the realization of a shock and smaller positions are needed. The model in the paper is qualitative, but a quantitative exploration in this sense may be interesting for future research.

History Dependence and Debt Dilution The strategies that support the efficient allocation are history dependent. In particular, the pricing function $\mathbf{q}$ does not only depend on the stock of outstanding debt and an indicator variable that records if the government has access to international credit markets or not as it is typically considered in the literature on quantitative sovereign default. The reason why history dependence is needed is connected to the debt-dilution problem. With long-term debt, any borrower's action that increases the likelihood of future outright default is tantamount to a (partial) default, because it imposes a capital loss to the holders of the outstanding debt. The equilibrium that implements the efficient allocation treats outright default and this more subtle partial default in a parallel fashion: The borrower is punished if he deviates from the path of plays by diluting existing debt too much, or if after a positive shock, he does not reduce his level of indebtedness.

This stands in contrast with standard sovereign debt models in which only outright default is punished with a trigger to autarky (with potential re-entry). This difference has important implications for how we think about the role of long-term debt, seniority, and pari passu clauses. Chatterjee and Eyigungor (2013) and Hatchondo et al. (2012) argue that within an equilibrium of the sovereign debt game in which only outright default is punished with trigger strategies, debt dilution is a problem and seniority clauses may be desirable. Moreover, absent rollover risk, short-term debt is very desirable and governments will opt to choose a maturity composition tilted toward short-term debt. This paper makes clear that such results are generated by the asymmetric treatment of outright default and dilution. In the best equilibrium of the sovereign debt game analyzed here, pari passu clauses and lack of seniority clauses are necessary for the best sustainable equilibrium to be equivalent to the efficient allocation. Hence, policies that introduce seniority may not be warranted.

Comparison with Alvarez and Jermann (2000) The implementation I propose, and its implications for the optimal maturity composition of debt, is applicable to environments other than the one considered here. For instance, it works for an economy with lack of commitment and public information[20] whenever the domestic agents discount more heavily than the international interest rate, $\beta < q$. Such condition arises in general equilibrium with a large number of countries, as shown in Alvarez and Jermann (2000). Under such condition, the efficient allocation is dynamic, in that $v'(v, z_H) \neq v'(v, z_L)$ for all v in the ergodic set.[21]

Alvarez and Jermann (2000) show that it is possible to implement the efficient allocation under lack of commitment with state contingent debt and endogenous debt limits. The main advantage of my implementation is that it neatly maps into the objects considered in applied works. In particular, it can be used to derive prices for defaultable bonds, and it has implications for the maturity composition of debt.

Finally, it is worth noticing that while the implementation I propose works in the environment considered by Alvarez and Jermann (2000), the converse is not true. Debt limits are not enough to implement the efficient allocation with private information. This follows from the fact that the efficient allocation with private information does not satisfy $qU'\left(c\left(z^t\right)\right) \geqslant \beta U'\left(c\left(z^t, z_{t+1}\right)\right)$ for all z^t, z_{t+1}. That is, the borrower is not always "bor-

[20]In this case, defaults will not be associated with ex post inefficiencies.

[21]This is not a property of the efficient allocation for an economy with no private information, lack of commitment (one sided), and $\beta = q$. In this case, because of backloading, the efficient allocation is converging to a steady state where the borrower's continuation value does not move with z.

rowing constrained." This observation follows from a version of the inverse Euler equation that holds in the economy considered here. Optimality requires that when the sustainability constraint does not bind, we have $qU'\left(c\left(z^{t-1}, z_H\right)\right) < \beta\mathbb{E}U'\left(c\left(z^{t-1}, z_H, z_{t+1}\right)\right)$. So, there is at least one state z_{t+1} for which $qU'\left(c\left(z^{t-1}, z_H\right)\right) < \beta U'\left(c\left(z^{t-1}, z_H, z_{t+1}\right)\right)$;that is, when current productivity is high, z_H, the borrower is "saving constrained." Thus, debt limits and Arrow-securities are not enough to decentralize the efficient allocation. After certain histories, a minimal asset holding requirements would be needed.

8 Final Remarks

In this paper, I show that key aspects of sovereign debt crises can be rationalized as part of the efficient risk-sharing arrangement between a sovereign borrower and foreign lenders in an economy with informational and commitment frictions. Along the outcome path that supports an efficient allocation, sovereign default episodes happen because of the need to provide incentives, despite being ex post inefficient.

This paper takes a first step toward bridging the gap between the literature on quantitative incomplete markets and the literature on optimal contracts. This paper is qualitative in nature; however, it shows that one can interpret the outcome of an optimal contracting problem through the lens of a standard sovereign debt game, and derive implications for interest rates and bond holdings, which are the focus of the applied literature. A quantitative evaluation of the model is a fruitful area for future research.

The implementation I propose—and its implications for the optimal maturity composition of debt—is applicable to environments other than the one considered here. As I mentioned, exact implementation may require very large positions, so it may be interesting to think about approximate implementation by imposing a cap on the debt positions. Moreover, it may be interesting to study the maturity composition in the best outcome of the sovereign debt game when the number of maturities available is smaller than the cardinality of the state space.

Finally, while the efficient allocation can be implemented as an equilibrium outcome of the sovereign debt game, the converse is not true. There is a *continuum* of equilibria and generically they are not efficient. Thus, despite the fact that agents are able to achieve the efficient outcome in a market setting, regulation by a supranational authority may indeed be helpful in avoiding inefficient equilibria.

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A Appendix

A.1 Reinterpretation: Taste Shock Economy

In this section, I show how I can reinterpret the environment in the text as an economy with two goods - a final and an intermediate good - where the stand-in domestic household is subject to a taste shock $\theta_t \in \Theta \equiv \{\theta_L, \theta_H\} = \{{z_H}^{1-\gamma}, z_L^{1-\gamma}\}$, which is *i.i.d.* over time and is privately observed by the domestic agent. The taste shock affects the domestic agent's marginal utility of consumption in a multiplicative fashion; a higher θ_t makes current consumption more valuable. A high taste shock corresponds to a low productivity shock in the original baseline economy. Intuitively, after either a high taste shock or a low productivity shock in the non-tradable sector, the marginal utility of imported intermediates used for domestic consumption is high. Define

$$\mathcal{C} = \frac{c}{z} \quad \text{and} \quad \theta = z^{1-\gamma} \tag{40}$$

where $\mathcal{C}$ can be thought as domestic consumption and θ is a taste shock. Let $\theta^t = (\theta_0, \theta_1, ..., \theta_t)$. Under (2), I can write the preferences for a stand-in domestic agent over a stochastic sequence $\{C_t(\theta^t)\}_{t=0}^{\infty}$ as

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t \Pr(\theta^t)\theta_t U\left(\mathcal{C}_t(\theta^t)\right) \tag{41}$$

From (RC), the resource constraint for this economy can be written as:

$$\mathcal{C}(\theta^t) + x(\theta^t) \leqslant f\left(m(\theta^{t-1})\right) \tag{RC}$$

where x are exports, as in the productivity shock formulation. An allocation for this taste shock economy is a stochastic process $\mathbf{x} \equiv \{m(\theta^{t-1}), \mathcal{C}(\theta^t), x(\theta^t)\}_{t=0}^{\infty}$. The allocation is feasible if it satisfies (RC) for all t, θ^t. Clearly, if $\mathbf{x}$ is feasible then $\mathbf{x}_z = \{m(z^{t-1}), \mathcal{C}(z^t)\theta_t^{1/(1-\gamma)}, y^*(z^t)\}_{t=0}^{\infty}$ is feasible for the baseline economy[22] and viceversa.

[22]This statement is true under the requirement that $x \geqslant 0$. In fact, in the taste shock economy, there is only one traded good, foreigners can in principle transfers resources to increase consumption of domestic households. The resource constraint is then

$$\mathcal{C} + x \leqslant f(m)$$

A.2 Omitted Proofs

A.2.1 Characterization of Solution:

I first establish some preliminary results. For notational convenience, the proofs in this Appendix use the taste shock notation. In particular, let

$$\theta_H = z_H^{1-\gamma} \text{ and } \theta_L = z_L^{1-\gamma}.$$

Note that under the assumption that $\gamma > 1$ we have that

$$\theta_H < \theta_L.$$

So the subscripts L and H stand in for low and high productivity shocks. Moreover, I let

$$\mathcal{C} = c/z.$$

Lemma 5 *Under Assumption 1: i) only local upward incentive compatibility constraints bind at a solution to (P); ii) if* $(m, \mathcal{C}(\theta), v'(\theta))$ *is incentive compatible and sustainable for* θ_L*, then it satisfies the sustainability constraint for all* $\theta \in \Theta$*.*

Proof. For part i) see Lemma 4 part i) in Thomas and Worrall (1990). For part ii), consider choosing $(\underline{u}, u(\theta), v'(\theta))$ instead of $(m, \mathcal{C}(\theta), v'(\theta))$, where $u(\theta) = U(\mathcal{C}(\theta))$. Let $(m, u(\theta), v'(\theta))$ be incentive compatible and such that $v'(\theta) \geqslant v_a$ for all θ. Furthermore, let it be such that it satisfies the sustainability constraint for θ_L in that:

$$\omega(\theta_L) \geqslant \theta_L U(f(m)) + \beta v_a$$

where $\omega(v, \theta) \equiv \theta u(\theta) + \beta v'(\theta)$. Consider two cases. First, if $U(f(m)) \geqslant u(\theta_L)$ then for all $\theta \in \Theta$ it follows that

$$\begin{aligned}
\omega(\theta) &\geqslant \omega(\theta_L) - (\theta_L - \theta)\, u(\theta_L) \\
&\geqslant \theta_L U(f(m)) + \beta v_a - (\theta_L - \theta)\, u(\theta_L) \\
&= \theta U(f(m)) + \beta v_a + (\theta_L - \theta)\left[U(f(m)) - u(\theta_L)\right] \\
&\geqslant \theta U(f(m)) + \beta v_a
\end{aligned}$$

In the 2-sector production economy, the resource constraint is

$$\frac{c}{z} + x \leqslant f(m) \quad \text{with } x \geqslant 0.$$

In characterizing the efficient allocation for the taste shock economy, I abstract from this constraint for simplicity. This is not affecting any of the results in the paper as this constraint is potentially binding outside the region of interest.

where in the first line I use the fact that $(m, u(\theta), v'(\theta))$ is incentive compatible; in the second line, the sustainability at θ_L; in the third line, I add and subtract $\theta U(f(m))$; and finally in the fourth line, I use the fact that $U(f(m)) \geqslant u(\theta_L)$. Then the sustainability constraint holds for all $\theta \in \Theta$. Now suppose that $U(f(m)) < u(\theta_L)$. In this case, the sustainability constraint is slack at θ_L, $\omega(\theta_L) > \theta_L U(f(m)) + \beta v_a$. Suppose for contradiction that there exists $\theta \in \Theta$ such that

$$\omega(\theta) \leqslant \theta U(f(m)) + \beta v_a$$

Then, notice that

$$\begin{aligned}
\omega(\theta_L) &\leqslant \omega(\theta) + (\theta_L - \theta)u(\theta_L) \\
&\leqslant [\theta U(f(m)) + \beta v_a] + (\theta_L - \theta)u(\theta_L) = \theta_L u(\theta_L) + \beta v_a + \theta\,[U(f(m)) - u(\theta_L)] \\
&< \theta_L u(\theta_L) + \beta v_a \leqslant \omega(\theta_L)
\end{aligned}$$

where the first line follows from incentive compatibility; in the second line, I use the contradiction hypothesis; in the third line, I used the fact that, by assumption, $U(f(m)) < u(\theta_L)$; and finally that $v(\theta_L) \geqslant v_a$. This delivers a contradiction and the result follows. Q.E.D.

Part i) states that the relevant incentive compatibility constraint is the one for which the borrower of type θ_H wants to report being of type θ_L. This result is standard; see for instance Thomas and Worrall (1990). Part ii) states that the only relevant sustainability constraint is the one for the low productivity shock type, θ_L. That is, the borrower is more tempted to exit the contract with the lenders when he values resources the most. This is not standard. Models with lack of commitment and no incentive problem typically display the opposite binding pattern.[23]

Consider now the solution for the problem in (P) where, in light of Lemma 5, we can drop the sustainability constraint for θ_L and the incentive compatibility constraint for θ_H. It is convenient to rewrite (P) using a change of variable: instead of $(m, \mathcal{C}(\theta), v'(\theta))$, consider choosing $(m, u(\theta), v'(\theta))$ instead. With this change of variable, the problem becomes

$$\text{(P')} \qquad B(v) = \max_{\underline{u}, u(\theta), v'(\theta)} f(m) - m + \sum_{\theta \in \Theta} \mu(\theta) \left[-C(u(\theta)) + qB(v'(\theta)) \right]$$

[23] In this particular application with taste shock the binding pattern of (SUST) when θ is observable depends on parameters. In an endowment economy – like the one considered in Kehoe and Levine (1993) and Alvarez and Jermann (2000) – the constraint (SUST) binds first for high output realizations. The argument in Lemma 5 is valid for those environments as well.

subject to

$$\theta_H u(\theta_H) + \beta v'(\theta_H) \geqslant \theta_H u(\theta_L) + \beta v'(\theta_L) \tag{42}$$

$$\theta_L u(\theta_L) + \beta v'(\theta_L) \geqslant \theta_L U(f(m)) + \beta v_a \tag{43}$$

$$v'(\theta) \geqslant v_a \quad \forall\theta \tag{44}$$

$$\sum_{\theta\in\Theta} \mu(\theta)\left[\theta u(\theta) + \beta v'(\theta)\right] = v \tag{45}$$

where $C : [U(0), U(\infty)] \to \mathbb{R}$ is $C = U^{-1}$.

Denote the decision rules associated with (P) as $m(v) : [v_a, \bar{v}] \to \mathbb{R}$ and $u(v,\theta), v'(v,\theta) : [v_a, \bar{v}] \times \Theta \to \mathbb{R}$. Moreover, define $\omega(v,\theta) \equiv \theta u(v,\theta) + \beta v'(v,\theta)$. The next lemmas establish some property of B. Q.E.D.

Lemma 6 *Under Assumptions 1–2,* B *is strictly concave and the maximizers* $m, v'(\theta), c(\theta) : [v_a, \bar{v}] \to \mathbb{R}$ *are continuous functions.*

Consider the problem (P′). Consider an further change of variable. Instead of $(m, u(\theta), v'(\theta))$, consider choosing $(\underline{u}, u(\theta), v'(\theta))$ where $\underline{u} = U(f(m))$. With this change of variable, the objective function becomes

$$H(\underline{u}) + \sum_{\theta\in\Theta} \mu(\theta)\left[-C(u(\theta)) + qB(v'(\theta))\right] \tag{46}$$

where $H(\underline{u}) \equiv f \circ \kappa(\underline{u}) - \kappa(\underline{u}) = C(\underline{u}) - \kappa(\underline{u})$ with $\kappa : [U(f(0)), U(f(m^*))] \to [0, m^*]$ is $\kappa = f^{-1} \circ C$ so that $\underline{u} = U(f(\kappa(\underline{u})))$, and the sustainability constraint (43) becomes

$$\theta_L u(\theta_L) + \beta v'(\theta_L) \geqslant \theta_L \underline{u} + \beta v_a.$$

With this formulation, the constraint set is linear in the choice variables, $(\underline{u}, u(\theta), v'(\theta))$. Under Assumption 2, H is concave. Therefore, by standard arguments, B is strictly concave. For the second part, just notice that the objective function in (46) is strictly concave and constraint set is convex; hence, its solution is unique. Thus, from the Theorem of the Maximum we have that $\underline{u}, u(\theta), v'(\theta) : [v_a, \bar{v}] \to \mathbb{R}$ are continuous in v. And so are $m, v'(\theta), c(\theta) : [v_a, \bar{v}] \to \mathbb{R}$. Q.E.D.

Differentiability of B can be established by applying the Benveniste and Scheinkman theorem, see Theorem 4.10 in Stokey et al. (1989). To this end, notice that imported intermediate goods and period utils are interior, in that $m > 0$ and $u(\theta) > U(0)$. The next Lemma establishes this result.

Lemma 7 *Under Assumptions 1–2, for all* $v \in (v_a, \bar{v})$*, if* $(m, u(\theta), v'(\theta))$ *is the solution to (P′) then* $m > 0$ *and* $u(\theta) > U(0)$.

Proof. Consider m first. Suppose, for contradiction, that $m = 0$. By Lemma 5 part ii), the relevant sustainability constraint is for type θ_L. There are two cases. If $\omega(\theta_L) > \theta_L U(f(0)) + \beta v_a$, then it is possible to increase m without violating the sustainability constraint and increasing the lenders' value, thus arriving at a contradiction. If, instead, $\omega(\theta_L) = \theta_L U(f(0)) + \beta v_a$, then the (45) implies that

$$\omega(\theta_H) > \theta_H U(f(0)) + \beta v_a$$

So it is possible to increase $u(\theta_L)$ by $\varepsilon > 0$, adjust $v'(\theta_H)$ and $u(\theta_H)$ so that (42) and (45) are satisfied

$$\begin{aligned} \theta_H \varepsilon_H + \beta \varepsilon_{VH} &\geqslant \theta_H \varepsilon \\ \mu(\theta_L)\theta_L \varepsilon + \mu(\theta_H)\left[\theta_L \varepsilon_{UH} + \beta \varepsilon_{VH}\right] &= 0 \end{aligned}$$

and increase m by ε_m defined as

$$\theta_L \varepsilon = \theta_L \left[U\left(f(\varepsilon_m)\right) - U(f(0))\right]$$

so that the variation satisfies (43). The change in the objective function is given by

$$\Delta B \approx \mu(\theta_L) C'(u_L)\varepsilon + \mu(\theta_H)\left[C'(u(\theta_H))\varepsilon_{UH} + qB'(v'(\theta_H))\varepsilon_{VH}\right] + \left[f'(0) - 1\right]\varepsilon_m > 0$$

because $\lim_{m \downarrow 0} f'(m) = \infty$, thus arriving at a contradiction.

For $u(\theta) > U(0)$, just note that, by Assumption 1, U is unbounded from below and bounded from above. Then it must be that $u(\theta) > U(0)$. Q.E.D.

Lemma 8 *Under Assumptions 1–2,* $B : [v_a, \bar{v}] \to \mathbb{R}$ *is differentiable*

For any $v > v_a$, let $x = (m, u(\theta), v'(\theta))$ be the solution that attains $B(v)$ in (P'). Consider a neighborhood of v_0, $D(v_0, \varepsilon) = (v_0 - \varepsilon, v_0 + \varepsilon)$ for some small $\varepsilon > 0$. Given the interiority of m and $u(\theta)$, define $\hat{x}(v) = (\hat{m}, \hat{u}(v, \theta), \hat{v}'(v, \theta))$ for any $v \in D(v_0, \varepsilon)$ as follows:

$$\hat{m}(v) = m + \frac{(v - v_0)/\mathbb{E}\theta}{U'\left(f(m)\right) f'(m)}, \quad \hat{u}(v, \theta) = u(\theta) + \frac{v - v_0}{\mathbb{E}\theta}, \quad \hat{v}'(v, \theta) = v'(\theta)$$

so that, by construction, $\hat{x}(v)$ is feasible in (P') for all $v \in D(v_0, \varepsilon)$ and $\varepsilon > 0$ sufficiently small. The fact that $\hat{x}(v)$ satisfies promise keeping and incentive compatibility is obvious.

For the sustainability constraint, notice that:

$$
\begin{aligned}
\theta U\left(f\left(\hat{m}(\nu)\right)\right)+\beta\nu_a &= \theta U\left(f(m)\right)+\int_0^{\frac{(\nu-\nu_0)/\mathbb{E}\theta}{U'(f(m))f'(m)}}\theta U'(f(m+x))f'(m+x)dx+\beta\nu_a \\
&\leqslant \theta U\left(f(m)\right)+\beta\nu_a+\frac{\theta U'\left(f(m)\right)f'(m)}{U'\left(f(m)\right)f'(m)}\frac{(\nu-\nu_0)}{\mathbb{E}\theta} \\
&= \theta U\left(f(m)\right)+\beta\nu_a+\frac{\theta\left(\nu-\nu_0\right)}{\mathbb{E}\theta}=\theta\hat{u}(\theta)+\beta\hat{\nu}'(\theta)
\end{aligned}
$$

where I use the fact that $U'(f(x))f'(x)$ is decreasing in x. Then $\hat{x}(\nu)$ is feasible in (P) for all $\nu\in D(\nu_0,\varepsilon)$ where $\varepsilon>0$ is sufficiently small. Then, define $\underline{B}:D(\nu_0,\varepsilon)\to\mathbb{R}$ as

$$
\underline{B}(\nu)=f\left(\hat{m}(\nu)\right)-\hat{m}(\nu)+\sum_{\theta\in\Theta}\mu(\theta)\left[-C(\hat{u}(\theta))+qB(\nu'(\theta))\right]
$$

Then $\underline{B}(\nu)$ is concave and differentiable in ν, for all $\nu\in D(\nu_0,\varepsilon)$, and $\underline{B}(\nu)\leqslant\hat{B}(\nu)\leqslant B(\nu)$ because $\hat{x}$ is feasible at ν and $\underline{B}(\nu_0)=\hat{B}(\nu_0)=B(\nu_0)$. Thus, the Benveniste and Scheinkman theorem applies: B is differentiable at ν_0 and

$$
B'(\nu_0)=\underline{B}'(\nu_0)=-\sum_{\theta\in\Theta}\mu(\theta)\frac{C'\left(u(\theta)\right)}{\mathbb{E}\theta}+\frac{1}{\mathbb{E}\theta}\frac{f'(m)-1}{U'\left(f(m)\right)f'(m)} \tag{47}
$$

This concludes the proof. Q.E.D.

To economize on notation, throughout the appendix I will let $C_i=C(\nu,\theta_i)$, $u_i=u(\nu,\theta_i)$ and $\nu'_i=\nu'(\nu,\theta_i)$ for $i=L,H$. Let λ_{ic}, λ_{sust}, and λ_{pkc} be the Lagrange multipliers on the incentive compatibility (42), the sustainability (43), and the promise keeping (45) constraints. The necessary and sufficient conditions for an interior solutions are:

$$
u_H:\quad -\lambda_{pkc}=-\frac{C'\left(u_H\right)}{\theta_H}+\frac{\lambda_{ic}}{\mu_H} \tag{48}
$$

$$
u_L:\quad -\lambda_{pkc}=-\frac{C'\left(u_L\right)}{\theta_L}-\frac{\lambda_{ic}}{\mu_L}\frac{\theta_H}{\theta_L}+\frac{\lambda_{sust}}{\mu_L} \tag{49}
$$

$$
\nu'_H:\quad -\lambda_{pkc}=\frac{q}{\beta}B'(\nu'_H)+\frac{\lambda_{ic}}{\mu_H} \tag{50}
$$

$$
\nu'_L:\quad -\lambda_{pkc}=\frac{q}{\beta}B'(\nu'_L)-\frac{\lambda_{ic}}{\mu_L}+\frac{\lambda_{sust}}{\mu_L} \tag{51}
$$

$$
m:\quad f'(m)-1=\lambda_{sust}\theta_L U'\left(f(m)\right)f'(m) \tag{52}
$$

and the envelope condition is

$$
B'(\nu)=-\lambda_{pkc} \tag{53}
$$

Moreover, from (52) and the definition of H we have that

$$\lambda_{sust} = h'(\omega_L)$$

where $h(\omega_L) \equiv H\left(\frac{\omega_L - \beta v_a}{\theta_L}\right)$ and recall that $\omega_L = \theta_L u_L + \beta v'_L$.

A.2.2 Proof of Proposition 1

Before I turn to prove Proposition 1, I characterize the optimal allocation at autarky:

Lemma 9 *At $v = v_a$ it must be that $u_L(v_a) = U(f(0))$, $v'_L(v_a) = v_a$, and if $\beta > q\theta_H/\mathbb{E}(\theta)$ then $u_H(v_a) < U(f(0))$, and $v'_H(v_a) > v_a$.*

Proof. Let $v = v_a$. For θ_L, it must be that $u_L(v_a) = U(f(0))$ and $v'_H(v_a) = v_a$.[24] In fact, to deliver the value of autarky in a sustainable way, it must be that for all θ:

$$\omega_L(v_a) = \theta u_L(v_a) + \beta v'_L(v_a) = \theta_L U(f(0)) + \beta v_a \tag{54}$$

Moreover, the sustainability and the incentive compatibility constraints imply that

$$\begin{aligned} \omega_H(v_a) &\geqslant \omega_L(v_a) - (\theta_L - \theta_H) u_L(v_a) \\ &= \theta_L U(f(0)) + \beta v_a - (\theta_L - \theta_H) u_L(v_a) \end{aligned} \tag{55}$$

Combining (54) and (55), it follows that

$$\begin{aligned} \theta_H U(f(0)) + \beta v_a &\geqslant \theta_L U(f(0)) + \beta v_a - (\theta_L - \theta_H) u_L(v_a) \\ &\iff (\theta_L - \theta_H) u_L(v_a) \geqslant (\theta_L - \theta_H) U(f(0)) \end{aligned}$$

which implies that $u_L(v_a) \geqslant U(f(0))$. Furthermore, (54) and the fact that $v'_L \geqslant v_a$ imply that $u_L(v_a) = U(f(0))$, as desired.

For θ_H, there are two possibilities: i) $u_H(v_a) < U(f(0))$ and $v'_H(v_a) > v_a$ or ii) $u_H(v_a) = U(f(0))$ and $v'_H(v_a) = v_a$. Assume that $\beta > q\theta_H/\mathbb{E}(\theta)$. Suppose by way of contradiction that we are in case ii). In this case v_a is an absorbing state and $B(v_a) = 0$. Consider a two-period variation that decreases current consumption after θ_H by ε and it increases it by ε/K for any state in the next period. That is:

$$\begin{aligned} \mathcal{C}_L &= f(0), \ \ v'_H = v_a \\ \mathcal{C}_H &= f(0) - \varepsilon, \ \ v'_L = \sum_i \mu_i \theta_i U(f(0) + \varepsilon/K) + \beta v_a \end{aligned}$$

[24]Note that this holds for all $\theta \in \Theta$ such that $\theta \neq \min_{\theta \in \Theta} \theta$.

for some $\varepsilon > 0$ and $K > 0$ such that the variation satisfies the promise keeping constraint at v_a in that:

$$
\begin{aligned}
(56) \qquad & \theta_L U(f(0)) + \beta v_a = U(f(0)-\varepsilon) + \beta \sum_i \mu_i \theta_i U(f(0)+\varepsilon/K) + \beta^2 v_a \\
& \iff [U(f(0)) - U(f(0)-\varepsilon)] = \beta \frac{\mathbb{E}(\theta)}{\theta_H} [U(f(0)+\varepsilon/K) - U(f(0))]
\end{aligned}
$$

To get a contradiction, it suffices to show that there is some $\varepsilon > 0$ and $K > 0$ such that (56) holds and

$$
(57) \qquad B(v_a) = 0 < \mu_H [\varepsilon - q\varepsilon/K] = \mu_H [1 - q/K]\,\varepsilon \iff K > q
$$

Rewrite (56) as

$$
\int_0^\varepsilon U'(f(0)-e)\,de = \frac{\beta}{K}\frac{\mathbb{E}(\theta)}{\theta_H} \int_0^\varepsilon U'(f(0)+e/K)\,de
$$

which implies, for $\varepsilon > 0$ sufficiently close to zero, that

$$
\begin{aligned}
K &= \beta \frac{\mathbb{E}(\theta)}{\theta_H} \left[\frac{\int_0^\varepsilon U'(f(0)+e/K)\,de}{\int_0^\varepsilon U'(f(0)-e)\,de} \right] \approx \beta \frac{\mathbb{E}(\theta)}{\theta_H} \left[\frac{U'(f(0))\,\varepsilon}{U'(f(0))\,\varepsilon} \right] = \beta \frac{\mathbb{E}(\theta)}{\theta_H} \\
&> \left(q \frac{\theta_H}{\mathbb{E}(\theta)} \right) \frac{\mathbb{E}(\theta)}{\theta_H} = q
\end{aligned}
$$

where in the last step I use the fact that we assumed that $\beta > q\theta_H/\mathbb{E}(\theta)$. Then (56) and (57) hold. This is a contradiction. Therefore, it must be that $u_H(v_a) < U(f(0))$ and $v'_L(v_a) > v_a$. Q.E.D.

The Lemma implies that autarky is not an absorbing state. When the borrower's value is equal to autarky if z_L is drawn then the borrower's consumption is equal to production in autarky, $f(0)$, and his continuation value is equal to autarky. When z_H is drawn, the borrower's valuation of current consumption is low. Therefore it is efficient to deliver the value of autarky, $U(z_H f(0)) + \beta v_a$, by providing lower consumption in the current period, $\mathcal{C}(v_a, \theta_L) < z_H f(0)$ (positive net export), and increasing the borrower's continuation value, $v'_H(v_a) > v_a$.

Proof of Proposition 1

Part i). First, let $v = v_a$. Combining the the sustainability constraint and the promise keeping constraint, it follows that

$$
\begin{aligned}
v_a &= v = \sum_i \mu_i \left[\theta_i u_i + \beta v_i'\right] \geqslant \sum_i \mu_i \left[\theta_i U\left(f(m)\right) + \beta v_a\right] \\
&> \sum_i \mu_i \left[\theta_i U\left(f(0)\right) + \beta v_a\right] = v_a \;\text{ if } m > 0
\end{aligned}
$$

Then it must be that $m(v_a) = 0$. For $v \in (v_a, \bar{v}]$, at an interior solution, (52) implies that $f'(m) - 1 = \lambda_{sust}\theta_L U'\left(f(m)\right) f'(m) \geqslant 0$. Hence $m \leqslant m^*$ and $m = m^*$ iff $\lambda_{sust} = 0$. First, I argue that if $\lambda_{sust}(v_1), \lambda_{sust}(v_2) > 0$ for $v_1 < v_2$, then it must be that $m(v_1) < m(v_2)$. Suppose for contradiction that $\lambda_{sust}(v_1), \lambda_{sust}(v_2) > 0$ and $m(v_1) \geqslant m(v_2)$. Then it must be that:

$$
\omega_L(v_1) \geqslant \omega_L(v_2), \;\; \lambda_{sust}(v_1) \leqslant \lambda_{sust}(v_2), \;\; \lambda_{pkc}(v_1) < \lambda_{pkc}(v_2) \tag{58}
$$

where $\lambda_{pkc}(v)$ is the Lagrange multiplier associated with the promise keeping constraint given promised utility v. By the envelope condition (53), this is equal to $-B'(v)$ and it is strictly increasing in v by strict concavity of B under Assumption 2. Rearranging (49) and (51) we can write that an interior optimum must satisfy:

$$
\frac{C'\left(u_L(v)\right)}{\theta_H} = \lambda_{pkc}(v) + \frac{\lambda_{sust}(v)}{\mu_L} - \frac{\lambda_{ic}(v)}{\mu_L}\frac{\theta_H}{\theta_L} = \Lambda(v) - \frac{\lambda_{ic}(v)}{\mu_L}\frac{\theta_H}{\theta_L} \tag{59}
$$

$$
-\frac{q}{\beta}B'(v_L'(v)) = \lambda_{pkc}(v) + \frac{\lambda_{sust}(v)}{\mu_L} - \frac{\lambda_{ic}(v)}{\mu_L} = \Lambda(v) - \frac{\lambda_{ic}(v)}{\mu_L} \tag{60}
$$

where I define $\Lambda(v) \equiv \lambda_{pkc}(v) + \lambda_{sust}(v)/\mu_L$. By (58), it follows that $\Lambda(v_2) > \Lambda(v_1)$. Now consider two cases. First, if

$$
\Lambda(v_1) - \Lambda(v_2) \geqslant \frac{\theta_H}{\theta_L}\left[\frac{\lambda_{ic}(v_1) - \lambda_{ic}(v_2)}{\mu_L}\right] \tag{61}
$$

then it follows that $\mathcal{C}_L(v_1) \geqslant \mathcal{C}_L(v_2)$. Moreover, $\theta_H/\theta_L = (z_H/z_L)^{1-\gamma} \in (0,1)$ and the term in square brackets is negative. It follows that

$$
\Lambda(v_1) - \Lambda(v_2) \geqslant \frac{\theta_H}{\theta_L}\left[\frac{\lambda_{ic}(v_1) - \lambda_{ic}(v_2)}{\mu_L}\right] > \frac{\lambda_{ic}(v_1) - \lambda_{ic}(v_2)}{\mu_L}
$$

Hence $v_L'(v_1) > v_L'(v_2)$. Notice that at an optimal solution, the relevant incentive compatibility constraint must hold with equality (see the discussion of part ii) below for a proof). Therefore, if the relevant incentive compatibility constraint binds for v_1, then the incentive compatibility constraint must not bind at the solution for v_2. In fact:

$$\begin{aligned} \omega_H(v_2) &> \omega_H(v_1) = \theta_H u_L(v_1) + \beta v'_L(v_1) \\ &> \theta_H u_L(v_2) + \beta v'_L(v_2) \end{aligned}$$

where the first inequality follows from the fact that $\omega_H(v_1) > \omega_H(v_2)$ and $v_2 > v_1$; the second follows from a binding incentive compatibility constraint for v_1; and the last follows from $u_L(v_1) \geqslant u_L(v_2)$ and $\omega_H(v_1) \geqslant \omega_H(v_2)$. This is a contradiction. Consider now the case in which (58) does not hold. This implies, together with (58) and (59), that $u_L(v_1) < u_L(v_2)$. Notice that a binding incentive compatibility constraint implies that

$$\omega_H(v) = \omega_L(v) - (\theta_L - \theta_H)u_L(v)$$

Using this equality in the promise keeping constraint, I can write:

$$\begin{aligned} v &= \mu_L\omega_L(v) + \mu_H\omega_H(v) = \omega_L(v) - \mu_H(\theta_L - \theta_H)u_L(v) \\ \iff & \omega_L(v) = v + \mu_H(\theta_L - \theta_H)u_L(v) \end{aligned}$$

Hence, the fact that $u_L(v_1) < u_L(v_2)$ implies that $\omega_L(v_1) < \omega_L(v_2)$, a contradiction.

I now turn to showing that there exists a v^* such that for all $v \geqslant v^*$, it must be that $m(v) = m^*$. Consider a relaxed version of (P') in which the sustainability constraint is dropped. In this relaxed problem, it can be shown that $\omega_L(v) \geqslant \theta_L(1-\beta)v + \beta v$. Hence, if $v \geqslant v^{**} \equiv [\theta_L U(f(m^*)) + \beta v_a] \,/\, [\theta_L(1-\beta) + \beta]$, the solution of this relaxed problem is a solution to the original problem. Thus, $m(v) = m^*$ for all $v \geqslant v^{**}$. Combining this with the fact that $m(v)$ is strictly increasing (and continuous) when λ_{sust} is binding, it follows that there must exist some $v^* \in (v_a, v^{**})$ for which $m(v) < m^*$ for all $v \in [v_a, v^*)$ and $m(v) = m^*$ for all $v \geqslant v^*$.

Part ii). First notice that the relevant incentive compatibility constraint must bind at an optimal solution. In fact, suppose by way of contradiction that it is slack. Then the optimality conditions imply that $v'_L \geqslant v'_H$ and $u_L > u_H$. Clearly this is not incentive compatible. Suppose for contradiction that $u_H \geqslant u_L$. For the relevant incentive compatibility constraint to be binding, it must be that $v'_L \geqslant v'_H$. By Lemma 9 below, we have that $v'_H > v_a$. Hence the solution is interior. Thus, I can combine the first order necessary conditions with respect to v'_L and v'_H, (50) and (51), to get

$$\frac{\lambda_{ic}}{\mu_H} \leqslant \frac{\lambda_{sust} - \lambda_{ic}}{\mu_L} < \frac{\lambda_{sust} - \lambda_{ic}\frac{\theta_H}{\theta_L}}{\mu_L} \tag{62}$$

where the last strict inequality follows from the fact that $\theta_H/\theta_L = (z_H/z_L)^{1-\gamma} \in (0,1)$.

Combining (62) with (48) and (49) implies that

$$\frac{C'(u_L)}{\theta_L} - \frac{C'(u_H)}{\theta_H} = \frac{\lambda_{sust} - \lambda_{ic}\frac{\theta_H}{\theta_L}}{\mu_L} - \frac{\lambda_{ic}}{\mu_L} > 0 \Rightarrow u_L > u_H$$

which is a contradiction. Hence, for all v it must be that $u_L(v) > u_H(v)$ or equivalently $C_L(v) > C_H(v)$. Consequently, incentive compatibility requires that $v'_H(v) > v'_L(v)$ for all v, as wanted.

Part iii). As in Thomas and Worrall (1990) Lemma 4 part (ii), suppose for contradiction that $b(v, \theta_H) < b(v, \theta_L)$ for some v. Then, consider offering the pooling allocation: $\hat{u}_H(v) = \hat{u}_L(v) = u_L(v)$ and $\hat{v}'_L(v) = \hat{v}'_H(v) = v'_L(v)$. Because the incentive compatibility constraint is binding at the optimal allocation, it follows that

$$\hat{\omega}(v, \theta_H) = \theta_H \hat{u}_H(v) + \beta \hat{v}'_H(v) = \theta_H u_L(v) + \beta v'_L(v) = \omega(v, \theta_H)$$

Hence, the promise keeping constraint is satisfied at the proposed solution. Incentive compatibility and sustainability are also trivially satisfied. Therefore, the proposed alternative pooling solution is feasible for v and is such that

$$\mu(\theta_H)\hat{b}(v, \theta_H) + \mu(\theta_L)\hat{b}(v, \theta_L) = b(v, \theta_L) > \hat{B}(v) = \mu(\theta_H)b(v, \theta_H) + \mu(\theta_L)b(v, \theta_L)$$

This is a contradiction. So, it must be that $b(v, \theta_H) \geqslant b(v, \theta_L)$. Suppose now that $b(v, \theta_H) = b(v, \theta_L)$. Then it must be that the pooling allocation is a solution to (P'). By part ii), the allocation is dynamic: $u_L(v) > u_H(v)$ and $v'_H(v) > v'_L(v)$, hence the pooling allocation cannot be a solution.

Part iv). The fact that the value of debt at autaky, $b(v, \theta) > 0$ follows from Lemma 9. Q.E.D.

A.2.3 Proof of Lemma 1

Proof of Lemma 1. Suppose for contradiction that B is (weakly) decreasing over $[v_a, \bar{v}]$ and so $v_a \in \arg\max_{v \in V} B(v)$. I am now going to show that a level of indebtedness strictly higher than $B(v_a)$ can be supported by delivering $v > v_a$, contradicting the fact that B is decreasing over its entire domain. Denote by x_a the allocation that attains $B(v_a)$. Consider the following variation for some $\varepsilon > 0$ sufficiently small:

$$m = \varepsilon > 0, \quad u(\theta) - u_a(\theta) = \varepsilon_u = U'(f(\varepsilon)) f'(\varepsilon)\varepsilon, \quad v'(\theta) = v'_a(\theta) \quad \forall \theta$$

By construction, the proposed variation satisfies the incentive compatibility and the sustainability constraints and attains a value for the borrower equal to $v_a + \varepsilon_u > v_a$. Thus, I am left to show that it increases the lenders' value too. The change in the lenders' value can be written as

$$\frac{\Delta B}{\varepsilon} \approx -\sum_{\theta\in\Theta} \mu(\theta)\left(\frac{C'\left(u_a\left(\theta\right)\right)}{C'\left(U'\left(f(\varepsilon)\right)\right)}\right) f'(\varepsilon) + \left[f'(\varepsilon)-1\right] = f'(\varepsilon)\left[1-\phi\right]-1$$

where

$$\phi \equiv \sum_{\theta\in\Theta} \mu(\theta)\left(\frac{C'\left(u_a\left(\theta\right)\right)}{C'\left(U'\left(f(\varepsilon)\right)\right)}\right) < 1$$

because from Lemma 9, it follows that $u_a(\theta) \leqslant U\left(f(0)\right) < U\left(f(\varepsilon)\right)$ and, in particular, $u_a(\theta_L) < U\left(f(0)\right)$. Thus, $\varepsilon > 0$ can be chosen to be sufficiently small that, by the Inada condition on f, $\Delta B/\varepsilon > 0$. Therefore it must be that $B(\nu_a + \varepsilon_u) \geqslant B(\nu_a) + \Delta B\varepsilon > B(\nu_a)$. Hence, B is not strictly decreasing, a contradiction. B is increasing in a neighborhood of ν_a. Moreover, B is strictly decreasing over $[\nu^*, \bar{\nu}]$. In fact, $\lambda_{sust}(\nu) = 0$ for $\nu \geqslant \nu^*$ and, therefore, $m(\nu) = m^*$. Then, from (47) it follows that for all $\nu \geqslant \nu^*$

$$B'(\nu) = -\sum_i \mu_i C'\left(u_i(\nu)\right) < 0$$

Thus, B is strictly decreasing over $[\nu^*, \bar{\nu}]$. The continuity and concavity of B imply that there exists $\tilde{\nu} \in (\nu_a\,, \nu^*)$ such that B is increasing for all $\nu \in [\nu_a, \tilde{\nu})$ and B is strictly decreasing over $[\tilde{\nu}, \bar{\nu}]$. Q.E.D.

A.2.4 Proof of Lemma 2

Consider any ν in the efficient region. Combining the envelope condition (53) with (51), I can write:

$$B'(\nu) \leqslant \frac{\beta}{q} B'(\nu) = B'(\nu_L') - \frac{\beta}{q}\left(\frac{\lambda_{ic} - \lambda_{sust}}{\mu_L}\right)$$

where I used the fact that $B'(\nu) \leqslant 0$ for all $\nu \in [\tilde{\nu}, \bar{\nu}]$ and that, by Assumption 1, $\beta/q < 1$. If $\lambda_{ic} > \lambda_{sust}$, I can rewrite the above inequality as

$$B'(\nu) \leqslant \frac{\beta}{q} B'(\nu) = B'(\nu_L') - \frac{\beta}{q}\left(\frac{\lambda_{ic} - \lambda_{sust}}{\mu_L}\right) < B'(\nu_L')$$

By concavity of B, it follows that $\nu_L'(\nu) < \nu\ \forall \nu \in [\tilde{\nu}, \bar{\nu}]$. Hence, it is sufficient to show that $\lambda_{ic} > \lambda_{sust}$.

Consider first the case in which there is partial insurance, i.e. $C'(u_L)/\theta_L \leqslant C'(u_H)/\theta_H$. Combine (48) and (49) to get

$$0 \geqslant \frac{C'\left(u_L\right)}{\theta_L} - \frac{C'\left(u_H\right)}{\theta_H} = \frac{1}{\mu_L}\left[\lambda_{sust} - \lambda_{ic}\frac{\theta_H}{\theta_L}\right] - \frac{1}{\mu(\theta_L)}\lambda_{ic}$$

Rearranging terms, I obtain

$$\lambda_{sust} \leqslant \lambda_{ic}\left(\frac{\mu_L}{\mu_H}+\frac{\theta_H}{\theta_L}\right)=\lambda_{ic}\left(\frac{\mathbb{E}\left(\theta\right)}{\mu_H\theta_L}\right)\leqslant \lambda_{ic}$$

where in the last step I use the fact that $\mu_H\theta_L \geqslant \mathbb{E}\left(\theta\right)$, which is true under Assumption 3.

Consider now the case with $C'(u_L)/\theta_L > C'(u_H)/\theta_H$.[25] For this case, I will use the following two results:

Claim 1 *If* $C'(u_L)/\theta_L > C'(u_H)/\theta_H$ *and*

$$-B'\left(\nu\right) \geqslant \frac{C'\left((1-\beta)\nu/\mathbb{E}\theta\right)}{\theta_L}-\left(\frac{\theta_L-\mathbb{E}\theta}{\mathbb{E}\theta}\right)h'\left(\nu+\frac{\theta_L-\mathbb{E}\theta}{\mathbb{E}\theta}(1-\beta)\nu\right) \tag{63}$$

then $\nu_L'\left(\nu\right)<\nu$.

Proof of Claim 1. Suppose by way of contradiction that $\nu_L'\left(\nu\right) \geqslant \nu$. Notice that the incentive compatibility constraint and the promise keeping constraint imply that

$$\begin{aligned}\nu &= \mu_L\left[\theta_L u_L+\beta\nu_L'\right]+\mu_H\left[\theta_H u_H+\beta\nu_H'\right]=\mu_L\left[\theta_L u_L+\beta\nu_L'\right]+\mu_H\left[\theta_H u_L+\beta\nu_L'\right]\\ &= \mathbb{E}\left(\theta\right)u_L(\nu)+\beta\nu_L'(\nu)\Rightarrow \nu_L'(\nu)=\frac{\nu-\mathbb{E}\theta u_L(\nu)}{\beta}\end{aligned}$$

Then under our contradiction hypothesis, $\nu_L'(\nu)\geqslant\nu$, it follows that

$$u_L<\frac{(1-\beta)\nu}{\mathbb{E}\theta}. \tag{64}$$

Combining the first order conditions (48) and (49) with the envelope condition I obtain

$$\begin{aligned}B'\left(\nu\right) &= -\frac{\sum_i \mu_i C'\left(u_i\right)}{\mathbb{E}\theta}+\frac{\theta_L}{\mathbb{E}\theta}h'\left(\omega_L\right)\\ &\geqslant -\frac{C'\left(u_L\right)}{\theta_L}+\frac{\theta_L}{\mathbb{E}\theta}h'\left(\omega_L\right)\\ &\geqslant -\frac{C'\left((1-\beta)\nu/\mathbb{E}\theta\right)}{\theta_L}+\frac{\theta_L}{\mathbb{E}\theta}h'\left(\omega_L\right)\end{aligned} \tag{65}$$

where the second line follows from $C'(u_L)/\theta_L \leqslant C'(u_H)/\theta_H$ and the third from (64) and the convexity of C. Moreover, combining the first order conditions (50) and (51) with the

[25]This case never arises in any of my numerical simulation.

envelope condition I obtain

$$
\begin{aligned}
(66)\ B'(v) &= \frac{q}{\beta}\sum_i \mu_i B'\left(v_i'\right) + h'(\omega_L) \\
&< B'(v) + h'(\omega_L) \\
&\leqslant -\frac{C'((1-\beta)v/\mathbb{E}\theta)}{\theta_L} + \left(\frac{\theta_L-\mathbb{E}\theta}{\mathbb{E}\theta}\right) h'\left(v + \frac{\theta_L-\mathbb{E}\theta}{\mathbb{E}\theta}(1-\beta)v\right) + h'(\omega_L)
\end{aligned}
$$

where the second line follows from $v_H' > v_L' \geqslant v \geqslant \tilde{v}$, concavity of B, and the fact that $q \geqslant \beta$; the third follows from condition (63). I can then combine (65) with (66) to obtain

$$
\left(\frac{\theta_L-\mathbb{E}\theta}{\mathbb{E}\theta}\right) h'\left(v + \frac{\theta_L-\mathbb{E}\theta}{\mathbb{E}\theta}(1-\beta)v\right) > \left(\frac{\theta_L-\mathbb{E}\theta}{\mathbb{E}\theta}\right) h'(\omega_L).
$$

Because of concavity of h (which in turn follows from concavity of H) the above condition holds if and only if

$$
v + \frac{\theta_L-\mathbb{E}\theta}{\mathbb{E}\theta}(1-\beta)v < \omega_L = v + (\theta_L-\mathbb{E}\theta)\,u_L \leqslant v + \frac{\theta_L-\mathbb{E}\theta}{\mathbb{E}\theta}(1-\beta)v
$$

where the last inequality follows from (64). Then I arrived at a contradiction. Therefore under the stated conditions it must be that $v_L'(v) < v$. □

Claim 2 *Under the technical assumption (14) and with $\mu_H\theta_L \geqslant \mathbb{E}\theta$, the value function B satisfies condition (63).*

Proof of Claim 2. To prove this claim, I use the fact that the operator $\mathbb{T}$ defined by the right side of (P′) is a contraction mapping. In particular, let the domain of $\mathbb{T}$ be $C^1([v_a, \bar{v}])$. Equipped with the norm $\|\cdot\|$ defined as $\|f\| = \|f\|_{\sup} + \|f'\|_{\sup}$ where $\|\cdot\|_{\sup}$ is the sup-norm, this space is a Banach space. Consider an arbitrary function $B_n \in C^1([v_a, \bar{v}])$ that satisfies (63). I next show that $B_{n+1} = \mathbb{T}B_n$ also satisfies the property (63). For a generic v, I consider two cases. First, suppose that $v_{Ln}' \geqslant v$. (subscript n denotes the policy function associated to the problem in (P′) using B_n as continuation value for the lenders.) By Claim 1 – which can be proved for a generic B_n that satisfies (63) – it follows that $C'(u_H)/\theta_H \geqslant C'(u_L)/\theta_L$. Under the assumption $\mu_H\theta_L \geqslant \mathbb{E}\theta$ this in turn implies that $\lambda_{ic,n} - h'(\omega_{Ln}) \geqslant 0$ and so from (51) and the envelope condition I obtain

$$
\begin{aligned}
-B_{n+1}'(v) &= -\frac{q}{\beta}B_n'\left(v_{Ln}'\right) + \frac{\lambda_{ic,n} - h'(\omega_{Ln})}{\mu_L} \\
&\geqslant -\frac{q}{\beta}B_n'\left(v_{Ln}'\right) \\
&\geqslant -B_n'(v) \\
&\geqslant \frac{C'((1-\beta)v/\mathbb{E}\theta)}{\theta_L} - \left(\frac{\theta_L-\mathbb{E}\theta}{\mathbb{E}\theta}\right) h'\left(v + \frac{\theta_L-\mathbb{E}\theta}{\mathbb{E}\theta}(1-\beta)v\right)
\end{aligned}
$$

where the second line follows from $\lambda_{ic,n} - h'(\omega_{Ln}) \geqslant 0$, the third from $v'_{Ln} \geqslant v$ and concavity of B_n, the fourth by the fact that B_n satisfies (63).

Second, suppose that $v'_{Ln} < v$. From (49) and the envelope condition, I obtain

$$-B'_{n+1}(v) = \frac{C'(u_L)}{\theta_L} + \frac{1}{\mu_L}\left(\lambda_{ic,n}\frac{\theta_L}{H} - h'(\omega_{Ln})\right)$$

Notice that since $v'_{Hn} > v'_{Ln}$, (50) and (51) imply that $\lambda_{ic,n} > \mu_H H'(\omega_{Ln})$. Using $\lambda_{ic,n} > \mu_H H'(\omega_{Ln})$ into the above relation we obtain

$$\begin{aligned} -B'_{n+1}(v) &> \frac{C'(u_L)}{\theta_L} - \frac{1}{\mu_L}\left(1 - \mu_H\frac{\theta_H}{\theta_L}\right)h'(\omega_{Ln}) \\ &> \frac{C'((1-\beta)v/\mathbb{E}\theta)}{\theta_L} - \left(\frac{1}{\mu_L} - \frac{\mu_H\theta_H}{\mu_L\theta_L}\right)h'\left(v + \frac{\theta_L - \mathbb{E}\theta}{\mathbb{E}\theta}(1-\beta)v\right) \\ &\geqslant \frac{C'((1-\beta)v/\mathbb{E}\theta)}{\theta_L} - \left(\frac{\theta_L - \mathbb{E}\theta}{\mathbb{E}\theta}\right)h'\left(v + \frac{\theta_L - \mathbb{E}\theta}{\mathbb{E}\theta}(1-\beta)v\right) \end{aligned}$$

where the second line follows from the fact that $v'_{Ln} < v$ implies that $u_L > (1-\beta)v/\mathbb{E}\theta$ and $\omega_{Ln} > v + \frac{\theta_L - \mathbb{E}\theta}{\mathbb{E}\theta}(1-\beta)v$ together with convexity of C and concavity of h.The last line follows from our technical assumption (14) which implies that $\left(\frac{1}{\mu_L} - \frac{\mu_H\theta_H}{\mu_L\theta_L}\right) < \left(\frac{\theta_L}{\mathbb{E}\theta} - 1\right)$. Then, the operator $\mathbb{T}$ preserves the property (63). By corollary of the Contraction Mapping Theorem (Corollary 3.1 in Stokey et al. (1989)) it then follows that the unique fixed point of $\mathbb{T}$, the value B, also satisfies property (63). □

Combining the two claims above immediately implies that $v'_L(v) < v$ when there is over-insurance in that $C'(u_L)/\theta_L > C'(u_H)/\theta_H$ or $U'(c_L) < U'(c_H)$. Therefore it must be that $v'_L(v) < v$ for all $v \in [\tilde{v}, \bar{v}]$ as desired. Q.E.D.

A.2.5 Proof of Proposition 3

To prove Proposition 3, I first establish the following Lemma that fully characterizes the law of motion for the continuation values after a high productivity shock.

Lemma 10 *Under Assumptions 1–3, the efficient allocation is such that:*

i) For all $v \in [v_a, \tilde{v})$, $v'_H(v) \geqslant \tilde{v}$ and $v'_H(\tilde{v}) > \tilde{v}$.

ii) If $\beta = q$, then $\forall v \in (\tilde{v}, \bar{v}]$, $v'_H(v) > v$.

iii) Instead, if $\beta < q$, then there exists $\bar{v}_q \in (\tilde{v}, \bar{v})$ such that for all $v > \bar{v}_q$, $v'_H(v) < v$.

Proof. **Part i)**. Consider first borrower values in the region with *ex-post* inefficiencies. Let $v \in [v_a, \tilde{v})$. In this interval, it must be that $v'_H(v) \geqslant \tilde{v}$. Suppose, by way of contradiction, that $v'_H(v) < \tilde{v}$. By Lemma 7, we know that $u_H > U(0)$. Consider then the following variation: decrease u_H by ε_c and increase v'_H by ε_v, for some $\varepsilon_v > 0$ sufficiently small and $\varepsilon_c(\varepsilon_v) > 0$, defined as the unique solution to

$$\theta_H (u_H - \varepsilon_c) + \beta v'_H + \varepsilon_v = \theta_H u_H + \beta v'_H$$

This variation is feasible for v in (P')[26] and has a positive effect on the objective function:

$$\Delta B(v) = \mu_H \left[qB \left(v'_H + \varepsilon_v \right) - qB \left(v'_H \right) + \varepsilon_c(\varepsilon_v) \right] > 0$$

because it decreases the cost of providing consumption today and it also increases the value of future transfers if $\varepsilon_v > 0$ is sufficiently small. This is a contradiction. Hence, it must be that $v'_H(v) \geqslant \tilde{v}$ for all $v \in [v_a, \tilde{v})$. Notice how this argument applies for all v. Hence, $v'_H(v) \geqslant v$ for all v.

Consider now $v \in [\tilde{v}, \bar{v}]$. Let λ_{ic} be the Lagrange multiplier on the incentive compatibility constraint. Combining (50) and the envelope condition (53), the intertemporal condition for v'_H can be written as:

$$B'(v) = \frac{q}{\beta} B(v'_H) + \frac{\lambda_{ic}}{\mu_H} \tag{67}$$

Consider first $v = \tilde{v}$. In this case, (67) can be written as:

$$0 = \frac{\beta}{q} B'(\tilde{v}) = B(v'_H(\tilde{v})) + \frac{\beta}{q} \frac{\lambda_{ic}}{\mu_H} > B(v'_H(\tilde{v}))$$

Then because of concavity of B it must be that $v'_H(\tilde{v}) > \tilde{v}$.

Consider now borrower values in $(\tilde{v}, \bar{v}]$. I have to consider two cases, $\beta = q$ and $\beta < q$.

Part ii). For $\beta = q$, (67) specializes to

$$B'(v) = B(v'_H) + \frac{\lambda_{ic}}{\mu_H} > B'(v'_H)$$

Then, by concavity of B, it follows that $v'_H(v) > v$ for all $v \in (\tilde{v}, \bar{v}]$.

Part iii). For $\beta < q$, rewrite (67) as:

$$B'(v) = \frac{q}{\beta} B(v'_H) + \frac{\lambda_{ic}}{\mu_H} = B'(v'_H(v)) + \frac{q - \beta}{\beta} B'(v'_H(v)) + \frac{\lambda_{ic}(v)}{\mu_H}$$

[26] Note that the same variation is not feasible for θ_H because it would violate the incentive compatibility constraint. For θ_L instead the proposed variation is actually relaxing a non-binding incentive constraint (type θ_H not reporting θ_L).

Then $v'_H(v) < v$ if and only if

$$-\frac{q-\beta}{\beta}B'(v'_H(v)) > \frac{\lambda_{ic}(v)}{\mu_H} \tag{68}$$

Suppose for contradiction that $v'_H(v) \geqslant v$ for all $v \in [v_a, \bar{v}]$. Then it must be that for all v, condition (68) does not hold and $B'(v) \geqslant B'\left(v'_H(v)\right)$. Therefore, it follows that

$$-\frac{q-\beta}{\beta}B'(v) \leqslant -\frac{q-\beta}{\beta}B'(v'_H(v)) < \frac{\lambda_{ic}(v)}{\mu_H}$$

Since $\lambda_{ic}(v)$ is bounded from above, $B'(v)$ is strictly decreasing for all $v \geqslant \tilde{v}$, and $\lim_{v\to\bar{v}} B'(v) = \lim_{c\to\infty} -1/U'(c) = -\infty$, for v sufficiently large it must be that $-\frac{q-\beta}{\beta}B'(v) > \frac{\lambda_{ic}(v)}{\mu_H}$. This is a contradiction. Then, for v sufficiently high, condition (68) is met. Denote by $\bar{v}_q \in (\tilde{v}, \bar{v})$ the smallest value of promised utility such that (68) holds for all $v > \bar{v}_q$.[27] By the above argument, such a $\bar{v}_q$ exists. Q.E.D.

Part i) states that if v is in the region with *ex-post* inefficiencies, it transits to the efficient region the first time that a high productivity shock is drawn, $v'_H(v) \geqslant \tilde{v}$ for all $v \in [v_a, \tilde{v}]$. For borrower values in the efficient region, I have to consider two cases. If $\beta = q$, part ii) establishes that $v'_H(v) > v$ for all v. This is because lenders and the sovereign borrower discount the future at the same rate and it is optimal for incentive provision to increase continuation utility after θ_H is drawn. Finally, if the borrower is more impatient than the lenders, $\beta < q$, part iii) states that there is a threshold, $\bar{v}_q$, after which it is optimal to have $v'_H(v) < v$. The relative impatience of the borrower eventually dominates the incentive benefits from backloading payments after θ_H.

I can then turn to the proof of Proposition 3:

Proof of Proposition 3. In light of Lemma 1, I can restrict attention to the compact set $[v_a, \bar{v}_q] \subset [v_a, \bar{v}]$. In fact, starting from any $v \in (\bar{v}_q, \bar{v}]$, the continuation utility is transiting to $[v_a, \bar{v}_q]$ in a finite number of periods because $v > v'_L(v) > v'_H(v)$ for all $v \in (\bar{v}_q, \bar{v}]$. To show that there exists a unique stationary distribution, I will show that the conditions in Theorem 12.12 in Stokey et al. (1989) are satisfied. In particular, I need to show that Assumption 12.1 in Stokey et al. (1989) is satisfied. To this end, define the transition $Q : [v_a, \bar{v}_q] \times \mathcal{B}\left([v_a, \bar{v}_q]\right) \to \mathbb{R}$ as

$$Q(v, A) = \sum_i \mu_i \mathbb{I}\left\{v'_i(v) \in A\right\}$$

[27]Notice that I only show that it exists a $\bar{v}_q$ such that $v'_L(v) > v$ for $v \in [\bar{v}_q, \bar{v})$ and $v'_L(\bar{v}_q) = \bar{v}_q$. I haven't shown that for all $v < \bar{v}_q$ it must be that $v'_L(v) > v$. It is however possible to define $v_q \leqslant \bar{v}_q$ as the largest v such that for all $v \leqslant v_q$ we have that $v'_L(v) > v$. The support of the limiting distribution (see the next proposition) is a subset of $[v_a, v_q] \subset [v_a, \bar{v}_q]$. In all of my numerical simulations I find that $v_q = \bar{v}_q$.

I need to show that there exists a mixing point $v \in [v_a, \bar{v}_q]$, $K \geqslant 1$, and $\varepsilon > 0$ such that $Q^K(v_a, [v, \bar{v}_q]) \geqslant \varepsilon$ and $Q^K(\bar{v}, [v_a, v]) \geqslant \varepsilon$. Consider $\tilde{v}$ as the mixing point. Because $v_L'(v) < v$ for all $v \geqslant \tilde{v}$, it follows that starting at $\bar{v}_q$ after a sufficiently long (but finite) string of realizations of low productivity, θ_L, the continuation utility transits to the region with *ex-post* inefficiencies. Thus for some finite K, $Q^K(\bar{v}, [v_a, \tilde{v}]) \geqslant \mu_L^K > 0$. Furthermore, by Lemma 10, $v_H'(v) \geqslant \tilde{v}$ for any v. Hence, starting from any v_a after drawing K realizations of θ_H, the continuation value is in the efficient region. Therefore $Q^K(v_a, [\tilde{v}, \bar{v}_q]) \geqslant \mu_H^K > 0$. Then just let $\varepsilon = \min\{\mu_H^K, \mu_L^K\}$. This shows that $\tilde{v}$ is a mixing point. Therefore, Theorem 12.12 in Stokey et al. (1989) applies and there exists a unique stationary distribution Ψ^* to which any efficient allocation converges. The fact that the stationary distribution is non-degenerate follows from Lemmas 2 and 10. Q.E.D.

A.2.6 Sufficient Condition for Assumption 4 Part i)

In this section, I provide sufficient conditions on primitives such that the efficient allocation satisfies part i) of Assumption 4. For the efficient allocation to satisfy this property, it is sufficient to show that for all v, $v_L' = v_L'(v) < v - k$ for some positive constant k. Or, equivalently,

$$B'(v_L') - B'(v) > K \tag{69}$$

for some $K > 0$. Before I provide assumption on primitives, I consider the case in which the efficient allocation displays partial insurance. That is, if the marginal utility of consumption is higher after the realization of a low productivity shock, $U'(c_L) \geqslant U'(c_H)$, or equivalently if the marginal cost of providing one unit of utility is higher after a high productivity shock, $C'(u_H)/\theta_H \geqslant C'(u_L)/\theta_L$.

Claim 3 *Under the assumptions of Lemma 2, if the efficient allocation displays partial insurance, in that $C'(u_H)/\theta_H \geqslant C'(u_L)/\theta_L$ or equivalently $U'(c_H) \leqslant U'(c_L)$, and the following stronger version of condition (13) holds,*

$$\frac{\mu_H \theta_L}{\mathbb{E}\theta} > 1 + (1 - \mu_H)\left(\frac{q}{\beta} - 1\right) \tag{70}$$

then (69) holds.

Proof. Combining the envelope condition (53) with (51) I obtain

$$
\begin{aligned}
(71) \qquad B'(v_L') &= \frac{\beta}{q}B'(v) + \frac{\beta}{q}\left(\frac{\lambda_{ic} - \lambda_{sust}}{\mu_L}\right) \\
&= B'(v) + \frac{\beta}{q}\left(\frac{\lambda_{ic} - \lambda_{sust}}{\mu_L}\right) - \left(1 - \frac{\beta}{q}\right)B'(v) \\
&> B'(v) + \left[\frac{\beta}{q}\left(\frac{\lambda_{ic} - \lambda_{sust}}{\mu_L}\right) - \left(1 - \frac{\beta}{q}\right)\lambda_{sust}\right]
\end{aligned}
$$

where the last inequality follows from $B' < \lambda_{sust}$. To prove the claim, it suffices to show that the term in square brackets is strictly positive, or equivalently

$$\lambda_{ic} > \left[1 + \mu_L\left(\frac{q}{\beta} - 1\right)\right]\lambda_{sust}$$

To this end, note that under partial insurance, I can combine (48) and (49) to get

$$\lambda_{ic} \geqslant \frac{\mu_H \theta_L}{\mathbb{E}\theta}\lambda_{sust}$$

Then, under (70), I obtain

$$\lambda_{ic} \geqslant \frac{\mu_H \theta_L}{\mathbb{E}\theta}\lambda_{sust} > \left[1 + \mu_L\left(\frac{q}{\beta} - 1\right)\right]\lambda_{sust}$$

as wanted. □

From (71), it is clear that in the region with ex-post inefficiency, to show that $v_L' + k < v$ it suffices to show that $\lambda_{ic} > \lambda_{sust}$ only if $\lambda_{sust}\left(1 - \frac{\beta}{q}\right) \approx 0$. Next, I provide two sets of conditions on primitives such that the efficient allocation satisfies part i) of Assumption 4. First, I show this is the case when β is sufficiently close to q:

Claim 4 *Under the assumptions of Lemma 2 and with the stronger version of condition (13),*

$$(72) \qquad \frac{\mu_H \theta_L}{\mathbb{E}\theta} > K,$$

if β is sufficiently close to q then (69) holds.

Proof. Consider the case $\beta = q$. In this case, the proof of Lemma 2 is also valid for $v < \tilde{v}$. Hence, for all $v > v_a$, it must be that $v_L'(v) < v$. Moreover, the condition (72) ensures that (69) holds implying that $v_L'(v) + k < v$ whenever $v_L'(v) \geqslant v_a$ does not bind when $\beta = q$. Since policies are continuous in β, this is also true for β less than q but sufficiently close to it. □

The next claims show that we can get the same result if the role of intermediate exports in production, $f(m^*) - f(0)$, is not too large. To this end, I specialize the production function F to be $F(m,\ell) = \left[\alpha m^{1-1/\sigma} + (1-\alpha)\ell^{1-1/\sigma}\right]^{\sigma/(1-\sigma)}$ for some $\sigma > 1$.

Claim 5 *Under the assumptions of Lemma 2, if α is sufficiently close to zero then (69) holds.*

Proof. Consider the case $\alpha = 0$. In this case, $\tilde{v} = v_a$ and so the proof of Lemma 2 is valid for all the domain of B. Moreover, $\lambda_{sust}(v) = 0$ for all $v > v_a$. Hence, the first order conditions for v'_L for all $v > v_a$ is

$$B'(v) = \frac{q}{\beta} B'(v'_L) - \frac{\lambda_{ic}}{\mu_L} + \eta \tag{73}$$

where $\eta \geqslant 0$ is the multiplier on $v'_L \geqslant v_a$. Then, whenever $v'_L > v$, since $B'(\cdot) < 0$ and $q \geqslant \beta$ and we have

$$B'(v) \leqslant B'(v'_L) - \frac{\lambda_{ic}}{\mu_L}$$

Letting $K = \min_v \lambda_{ic}(v) > 0$, it follows that (69) holds. Since policies are continuous in α, this is also true for $\alpha > 0$ but sufficiently small. □

A.2.7 Proof of Lemma 4

Let Q_L be the space of bounded functions $q_L : [v_a, \bar{v}] \to [0, q/(1-q)]$ and let $\mathbb{T} : Q_L \to Q_L$ be defined by the right hand side of (31). That is:

$$(\mathbb{T} q_L)(v) = \begin{cases} q \sum_{i=L,H} \mu_i \left[1 + q_L(v'_i(v))\right] & \text{if } v \in (v_r, \bar{v}] \\ \frac{q}{1-q} \bar{R} & \text{if } v \in [v_a, v_r] \end{cases}$$

$\mathbb{T}$ satisfies the Blackwell's sufficient condition for a contraction mapping, see Theorem 3.3 in Stokey et al. (1989). Then, by the contraction mapping theorem, there exists a unique fixed point of $\mathbb{T}$, $\bar{q}_L$. To see that $\bar{q}_L$ is increasing, notice that $\mathbb{T}$ maps increasing functions into increasing functions. Then, by a corollary of the contraction mapping theorem (see Corollary 3.1 in Stokey et al. (1989)), it must be that $\bar{q}_L$ is increasing. Also notice that $\bar{q}_L(v) < q/(1-q)$ for all $v \in [v_a, \bar{v}]$ since there is always a strictly positive probability of the continuation value reaching v_a after a sufficiently long string of low productivity shocks. In the case considered here with no randomization, I cannot prove that $\bar{q}_L$ is strictly increasing, as it may have flat portions. To prove that $\bar{q}_L(v'_H(v)) > \bar{q}_L(v'_L(v))$, notice that the fact that $\bar{q}_L$ is increasing and $v'_H(v) > v'_L(v)$ imply that $\bar{q}_L(v'_H(v)) \geqslant \bar{q}_L(v'_L(v))$. Suppose, by way of contradiction, that there exists a v such that $\bar{q}_L(v'_H(v)) = \bar{q}_L(v'_L(v))$. This and the fact that $v'_H(v) > v > v'_L(v)$ requires that $v'_H(v), v$, and $v'_L(v)$ belong to a subset $[v_n, v_{n+1}]$ where $\bar{q}_L$ is constant. This in turn implies that for all $\tilde{v} \in [v_n, v_{n+1}]$ it must be that $\nu'_H(\tilde{\nu}), \nu'_L(\tilde{\nu}) \in [\nu_n, \nu_{n+1}]$. This is a contradiction because $\nu'_H(\nu_{n+1}) > \nu_{n+1}$ and, thus ,it does not belong to the set.[28] Hence, it must be that $\bar{q}_L(\nu'_H(\nu)) > \bar{q}_L(\nu'_L(\nu))$. Q.E.D.

A.3 Data Appendix

Data Description I consider the same 23 default events as Mendoza and Yue (2012): Argentina (1982, 2002), Chile (1983), Croatia (1992), Dominican Republic (1993), Ecuador (1999), Indonesia (1998), Mexico (1982), Moldova (2002), Nigeria (1983, 1986), Pakistan (1998), Peru (1983), Philippines (1983), Russia (1998), South Africa (1985, 1993), Thailand (1998), Ukraine (1998), Uruguay (1990), Uruguay (2003), Venezuela (1995), Venezuela (1998).

Annual data for GDP and consumption are gathered from the World Development Indicators (WDI). These are measured in real US dollars. As in Mendoza and Yue (2012), imported intermediates are the sum of categories for intermediate goods based on the Broad Economic Category (BEC) classification. The categories for intermediate goods are: (111) Food and beverages, primary, mainly for industry, (121) Food and beverages, processed, mainly for industry, (21) Industrial supplies not elsewhere specified, primary, (22) Industrial and lubricants, processed, (other than motor spirit), (42) Parts and accessories of capital goods (except transport equipment), (53) Part and accessories of transport equipment. For years 1962 through 2000, data is available from Feenstra et al. (2005) but is classified using the Standard International Trade Classification, revision 4 (SITC4). I use UN concordances to map SITC4 into BEC codes. For years 1976 through 2010, data is available through the World Bank's World Integrated Trade Solution (WITS) database, which has information from the UN's Comtrade database. This database provides the series for the above BEC codes when available. When I have data from both sources, I use the WITS data, which does not rely on the concordances. For years in which both sources provide data, I have cross referenced the values. Although the levels are not exactly the same, deviations from the trend (my variable of interest) are very similar across the two sources. I deflate the intermediate import data using the US producer price index (PPI) from the Bureau of Economic Analysis (BEA). Each annual series is logged and HP-filtered with a smoothing parameter of 100.

Default and drop in import of intermediates Here I document that the drop in import of intermediates is larger during default episodes than in recessions of similar magnitudes. To document this fact, I regress imported intermediates at time t on a constant,

[28] In the proof, I consider a set $[\nu_n, \nu_{n+1}]$ closed to the right. If the set is open to the right, $[\nu_n, \nu_{n+1})$, the same contradiction argument goes through because $\nu'_H(\cdot)$ is continuous and $\nu'_H(\nu) > \nu$ over $[\nu_a, \bar{\nu}]$ and thus $\nu'_H(\nu) - \nu > \varepsilon$ for all ν and some $\varepsilon > 0$ sufficiently small.

Table 1. OLS Regression: Intermediate Imports at time t

Variable	Coefficient Estimate	Standard Error
Constant	0.007	0.960
GDP at t	1.810	0.145
Default at t	-0.119	0.044
Default at $t-1$	-0.108	0.044
Default at $t-2$	-0.040	0.044
Default at $t-3$	-0.005	0.043
R^2=0.225; Number of observations = 714		
Intermediate imports and GDP are logged and HP-filtered.		

GDP at time t, and dummy variables that take value of one if there is a default in the country at time t, $t-1$, $t-2$ and $t-3$ from 1962 to 2010 for the 18 countries in the sample for which I have data on intermediate imports. The result for this simple regression are reported in Table 1 . The drop in intermediate inputs in the year of and the year following a sovereign default is more than 10 percent larger than what one would expect from a drop in output of the same magnitude, absent default.